NOTES ON TAME VALUED FIELDS

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A field K equipped with a (Krull) valuation v is **henselian** if it satisfies "Hensel's Lemma": every simple zero in the residue field of a monic polynomial with coefficients from the valuation ring, lifts to a zero. The study of henselian valued fields unites model theory and arithmetic geometry, relying heavily on classical and modern ramification theory, and has increasingly apparent links to tropical geometry. The original Ax–Kochen/Ershov Theorem ([AK65a, AK65b, Eršo65]) states that the complete first-order theory of a henselian valued field of residue characteristic zero (in a language suitable for valued fields) is obtained simply by specifying the theories of its value group and residue field. Similar results hold in the finitely ramified ([AK65b, Eršo65, Zie72, Bél99, ADJ24, LL21]) and (separably) algebraically maximal Kaplansky ([Del82]) settings.

A "typical" AKE principle is then a statement for a class $\mathcal{C}$ of valued fields, possibly enriched, which states that the theory of an element K of $\mathcal{C}$ is determined, modulo the common theory of $\mathcal{C}$, by the theories of the value group and residue fields of K, again possibly enriched, and in suitable languages.

The class of tame valued fields is one of the most general settings in which these kinds of principles have been proved. A valued field (K, v) is **tame** if (Definition 1.7) it is henselian, and every finite extension is itself tame, i.e. the degree n of the extension is equal to the product of the ramification degree e and the degree f of the separable part of the residue extension, $n = ef$ holds, and e is coprime to the residue characteristic. Kuhlmann gave a simple characterization ([Kuh16, Theorem 3.2]): (K, v) is tame if and only if it is algebraically maximal and has a p-divisible value group and perfect residue field, where p is the residue characteristic exponent.

There are numerous examples and non-examples:

Example. Algebraically closed valued fields are tame, and Hahn series fields $k((t^\Gamma))$, with k perfect and Γ p-divisible, are tame. Fields $k((t))$ of formal power series are tame if and only if $\mathrm{char}(k) = 0$.

The theory of tame valued fields, due in the main to F.-V. Kuhlmann and coauthors, is one of the most general settings in which Ax–Kochen/Ershov principles are known. In the past years, the algebraic and model theoretic study of tame valued fields has attracted enormous interest, for example see [AB26, AF16, AJ18, AJ22, AJ24, AK16, BK17, Daa24, BDM25, Jah24, JK25, JS20, JvdS26, Kar23, Kar24, Ket26, KR23, Sin22, vdS25] among others.

The principal obstruction to the model theory of valued fields in positive characteristic is the phenomenon of defect: this is the quantity d in the equation $[L : K] = d\,e\,f$, for henselian K, with d a power of p (Ostrowski), see Appendix B.

Example. Over $\mathbb{F}_p((t))^{\mathrm{perf}}$, the Artin–Schreier extension given by $\wp(\vartheta) = 1/t$ is immediate of degree p. So this field is not tame.

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The two principal structural results in Kuhlmann's tame theory are the following, due to Kuhlmann and his collaborators:

Theorem 1 (Generalized Stability, Kuhlmann)**.** *Let $(F/K, v)$ be a valued function field without transcendence defect, and suppose that (K, v) is defectless. Then (F, v) is defectless.*

Theorem 2 (Henselian rationality, Kuhlmann)**.** *Let $(F/K, v)$ be an immediate henselized function field of dimension one, and suppose that (K, v) is tame. Then $(F/K, v)$ is "henselian rational".*

These have themselves been recently reproved in [DHLY24, DL26] by new methods, combining tools of model theory and Berkovich geometry. The original arguments are understandably complicated, and are spread over at least [KK05, Kuh10a, Kuh10b, Kuh16, Kuh19a, Kuh19b, KV14, KP16]. Generalized stability is essentially an extension of Grauert–Remmert stability, given in the analytic case, which has an analytic proof in [Tem10, §6.3]. Henselian rationality is an extension of Epp's elimination of wild ramification [Epp73] (corrected in [Kuh03]).

In this note we give a compressed exposition (in which we prove no new theorems) of Theorems 1 and 2.

It is these structural results that yield the embedding principles and model theoretic results of [Kuh16, KP16, Ans25]. In the final section we reprove the Embedding Lemma for the theory of separably tame valued fields and deduce the standard Ax–Kochen/Ershov style consequences for tame valued fields.

Theorem 3 (Tame valued fields, Kuhlmann)**.**

- **(i)** *The theory* $\mathbf{TVF}^{\mathrm{eq}}$ *of tame valued fields of equal characteristic satisfies the Ax–Kochen/Ershov principles* $\mathrm{AKE}^{\square}$*, for* $\square \in \{\equiv, \equiv_\exists\}$.
- **(ii)** *The theory* **TVF** *of tame valued fields satisfies the Ax–Kochen/Ershov principles* $\mathrm{AKE}^{\square}$*, for* $\square \in \{\preceq, \preceq_\exists\}$.

The structure of the proofs. Both Theorem 1 and 1 have a common engine: ramification analyses of Galois extensions of degree p, of valued function fields over tame fields, studied through "simplified minimal polynomials" and strong bases (Proposition 2.14). Of course there is a case distinction into Artin–Schreier extensions in equal characteristic and Kummer extensions in mixed characteristic.

For Theorem 1, there are three reductions lead to the core case, denoted ($\mathbf{H_{AC}}$), which assumes that K is algebraically closed, of rank 1, and of transcendence degree 1. These reductions, due to Kuhlmann, are also sketched in [Tem13, §2.1]. In the core case, there is a standard argument (**(D3)**) to reduce defectlessness to algebraic maximality. The ramification analysis is Proposition 2.24, and it shows that that degree-p steps over strongly inertially generated (strongly inertially generated) function fields fields are defectless (Lemma 2.27).

For Theorem 2, the notion of a strong basis is replaced by that of a "frame". The normal forms then appearing in the ramification analysis (Proposition 3.12), yield $E = K(\vartheta)^{\mathrm{h}}$, i.e. "henselian rationality". then the Main Proposition (Proposition 3.16), then the reduction from rank 1 to arbitrary rank.

Figure 1 is an illustration of the different cases involved the ramification analyses, working behind-the-scenes in Theorems 1 and 2. The columns are divided into three cases (non-exhaustive) describing the extension of a valuation to a rational function field. These are value transcendental ((VT)), residue transcendental ((RT)), and immediate ((IMM)). The rows correspond to equal characteristic ((EQ)) and mixed characteristic ((MIX)). Observe that the outcomes (SPLIT) and (RT-BASE) occur only in (EQ).

What these notes do . . . and what they do not do. I have endeavored to give a compressed exposition, proving no new theorems. My principal goal was to make the argument as self-contained as reasonably possible. I have allowed anything I deemed as "standard", which could be found in textbooks on valuation theory or commutative algebra (including [EP05, Kap42, KPR86, Bou89]), as well as selected other ingredients. Notably I include a list of more advanced "standard facts", listed as **(D1)**–**(D6)** in section 2.5.

There are numerous inter-dependencies between these results: Section 2 leads to Section 3, which reuses some of the earlier degree-p analysis. Both of course lead to the final section, which outlines the main embedding lemma suitable for separably tame vlaued fields. I also include three appendices. In Appendix A (Frobenius-closed bases), I reprove Kuhlmann's work on Frobenius-closed bases of function fields; this material is used only for the existence of strong bases (Proposition 2.14). Appendix B is simply a short proof of the Fundamental Equality (as opposed to just the Fundamental Inequality), which is very well-known, but a little bit lacking in the literature. Finally, Appendix C contains some further results on strong bases, almost entirely unused in any proof in the paper (except in 3.10).

These notes do not prove any significant new result. Let me again say that these results are work of Kuhlmann and various coauthors.

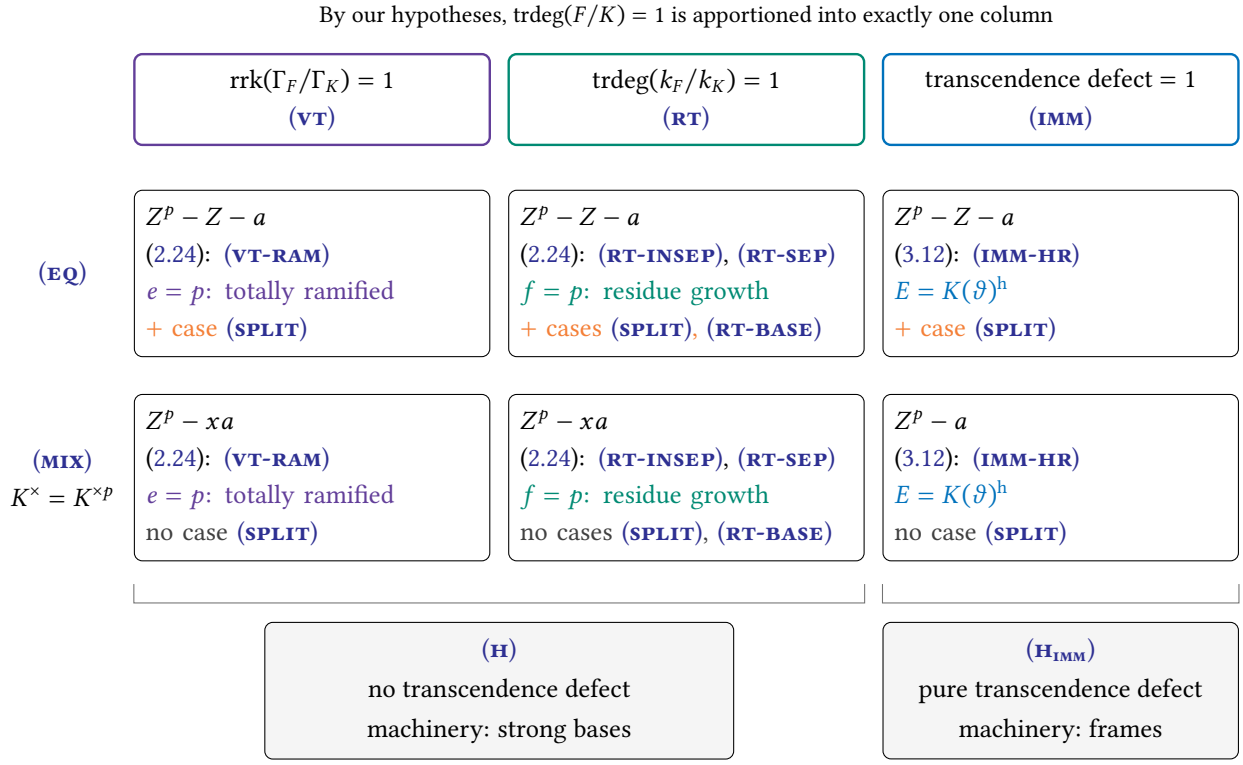


FIGURE 1. The case distinctions: two cases by characteristic versus three cases by the valuation extension F/K

How to read this paper. If you are familiar with this subject, as I expect you are, then you will probably want to skip Section 1, returning just when needed. If you are interested in the Generalized Stability Theorem, you are looking for Section 2. If you are interested in Henselian Rationality, you are looking for Section 3. If you are interested in the model theory of Separably Tame Valued Fields, you are looking for Section 4.

1. Selected ingredients, notation, and conventions

1.1. **Notation and invariants.** Throughout we consider valued fields (K, v) of residue characteristic $p > 0$. Our notation will usually suppress the name of the valuation, since we will not often be considering different valuations on the same field. Thus we denote by k_K the residue field, by Γ_K the value group, by $\mathcal{O}_K$ the valuation ring, and by $\mathfrak{m}_K$ the valuation ideal of (K, v). The residue of an element $x \in \mathcal{O}_K$ will usually be written as $\overline{x}$, or less often as $\mathrm{res}(x)$ when the simpler notation would be ambiguous. An extension $(L/K, v)$ of valued fields induces extensions k_L/k_K and Γ_L/Γ_K. Important quantities attached to $(L/K, v)$ include the **inertia degree** $f(L/K, v) := [k_L : k_K]$, the **ramification degree** $e(L/K, v) := (\Gamma_L : \Gamma_K)$, and the **defect** $d(L/K, v) := [L^h : K^h]/(e(L/K, v)f(L/K, v))$ for finite L/K, where $L^h = LK^h$ is the henselization of (L, v) (see [Kuh11, Theorem 4.14]). The defect $d(L/K, v)$ is always a power of p, by Ostrowski's lemma [Rib68, Théorème 2, p. 236] (see also [Kuh11, p. 300]). Since henselizations are immediate [EP05, Theorem 5.2.5], $d(L/K, v) = d(L^h/K^h, v)$. There are two important formulas governing extensions: the **Fundamental Equality** (Proposition B.1)

$$[L : K] = \sum_{w \supseteq v} e(L/K, w)f(L/K, w)d(L/K, w),$$

the sum running over prolongations of v to L, and the **Abhyankar Inequality** ([Bou89, VI §10.3])

$$\mathrm{trdeg}(L/K) \geq \mathrm{trdeg}(k_L/k_K) + \mathrm{rrk}(\Gamma_L/\Gamma_K),$$

where the **rational rank** $\mathrm{rrk}(G)$ of an abelian group G is the $\mathbb{Q}$-linear dimension of $G \otimes_{\mathbb{Z}} \mathbb{Q}$. The **transcendence defect** of $(L/K, v)$ is $\mathrm{trdeg}(L/K) - \mathrm{trdeg}(k_L/k_K) - \mathrm{rrk}(\Gamma_L/\Gamma_K)$, so an extension is **without transcendence defect** when the transcendence defect is zero. We say that (K, v) is **[separably] defectless** if every finite [separable] extension L/K satisfies $[L : K] = \sum_{w \supseteq v} e(L/K, w)f(L/K, w)$. Equivalently, $d(L/K, w) = 1$ for all such L and w, by the Fundamental Equality above. A valued field is **[[separably] algebraically]**

maximal if it admits no proper immediate [[separably] algebraic] extension. This notion of maximality is apparently due to F. K. Schmidt, but appears first in [Kru32, p. 191], see [Kap42, §1].

Defect is multiplicative in towers, i.e. $d(M/K,v) = d(M/L,v)\cdot d(L/K,v)$ for a tower of finite extensions $M/L/K$: when K is henselian this follows from the Fundamental Equality, since degree, ramification degree and inertia degree are all multiplicative in towers, and in general one may pass to henselizations, using $M^{\mathrm{h}} = ML^{\mathrm{h}}$ and $L^{\mathrm{h}} = LK^{\mathrm{h}}$.

For an extension $(L/K,v)$ of valued fields, we say that a family $(b_i)_{i\in I}$ in L is **valuation independent** over (K,v) if $v\big(\sum_{i\in J} c_i b_i\big) = \min_{i\in J} v(c_i b_i)$ for all finite $J \subseteq I$ and all coefficients $c_i \in K$. Such a family is necessarily K-linearly independent. A **valuation basis** of $(L/K,v)$ is a valuation independent K-linear basis of L. Valuation independence will appear later as condition (SB-1) in the definition of a strong basis (Definition 2.6), and other names for it in the literature are listed just before that definition.

Observation 1.1. Let $(L/K,v)$ be an extension of valued fields, and for $y \in L$ put $v(y-K) := \{v(y-c) \mid c \in K\}$.

(i) $(L/K,v)$ is immediate if and only if $v(y-K)$ has no largest element for every $y \in L\setminus K$.

(ii) If $b_1 = 1, b_2, \dots, b_n$ is a valuation basis of $(L/K,v)$, then the largest element of $v(y-K)$ is $v(y-c_1)$, for every $y = \sum_i c_i b_i \in L$. A valuation basis containing 1 exists if (K,v) is henselian and $[L:K] = e(L/K)f(L/K)$.

Proof. For **(i)**: suppose that $(L/K,v)$ is immediate, let $y \in L\setminus K$, and let $c \in K$. Immediacy gives us an element $d \in K^\times$ with $v(d) = v(y-c)$, and then $a \in \mathcal{O}_K$ with $\overline{a} = \overline{(y-c)/d}$. So $v\big((y-c)/d - a\big) > 0$, i.e. $v\big(y-(c+da)\big) > v(y-c)$ with $c+da \in K$. Conversely, let $y \in L\setminus K$. As $v(y-0)$ is not the largest element of $v(y-K)$, there is $c \in K$ with $v(y-c) > v(y)$. Then $v(y) = v(c) \in \Gamma_K$, and if $v(y) = 0$ then $\overline{y} = \overline{c} \in k_K$.

For **(ii)**: for an element $c \in K$ we have $v(y-c) = \min\{v(c_1-c), v(c_2b_2), \dots, v(c_nb_n)\} \le v(y-c_1)$. For the last claim, choose $\omega_1 = 1, \omega_2, \dots, \omega_f \in \mathcal{O}_L$ whose residues form a basis of k_L/k_K, and $\pi_1 = 1, \pi_2, \dots, \pi_e \in L$ whose values represent Γ_L/Γ_K. By [EP05, Lemma 3.2.2], the $ef = [L:K]$-many products $\omega_i\pi_j$ are valuation independent, so they form a valuation basis containing $\omega_1\pi_1 = 1$. □

So an element of an immediate extension has no best approximation in the base, whereas every element of a finite defectless extension of a henselian field has one. These classical facts are [CKR20, Lemma 2.6, Theorem 2.10 and Corollary 2.12], cf. [Kap42, Theorem 1] and [KV14, Lemma 2.1]. With the Fundamental Equality, **(ii)** shows that an extension of degree p of a henselian field in which some element has no best approximation is immediate [Kuh10b, Lemma 2.21(1)].

1.2. **Monomial valuations.** For a tuple $x = (x_1, \dots, x_r)$ and $\mu \in \mathbb{N}^r$ we write $x^\mu := x_1^{\mu_1}\cdots x_r^{\mu_r}$, and $K(x)$ and $K[x]$ for $K(x_1,\dots,x_r)$ and $K[x_1,\dots,x_r]$, respectively. For a tuple $y = (y_1,\dots,y_s)$ of elements of $\mathcal{O}_K$, we write $\overline{y} := (\overline{y_1},\dots,\overline{y_s})$.

Lemma 1.2 ([Bou89, VI §10.3, Theorem 1], [EP05, Theorem 3.4.2])**.** *Let $(L/K,v)$ be an extension of valued fields, and let $t = (t_1,\dots,t_r)$ and $u = (u_1,\dots,u_s)$ be tuples in $L^\times$ such that $v(t_1),\dots,v(t_r)$ are rationally independent over Γ_K, $v(u_j) = 0$ for all j, and $\overline{u_1},\dots,\overline{u_s}$ are algebraically independent over k_K. Then $t_1,\dots,t_r,u_1,\dots,u_s$ are algebraically independent over K, and*

$$v\Big(\sum_{\mu,\nu} c_{\mu\nu}\, t^\mu u^\nu\Big) = \min_{\mu,\nu} v(c_{\mu\nu}\, t^\mu u^\nu) = \min_{\mu,\nu}\Big(v(c_{\mu\nu}) + \sum_i \mu_i v(t_i)\Big)$$

for all coefficients $c_{\mu\nu} \in K$. Moreover $\Gamma_{K(t,u)} = \Gamma_K \oplus \bigoplus_i \mathbb{Z}v(t_i)$ and $k_{K(t,u)} = k_K(\overline{u})$.

In particular, the valuation on $K(t,u)$ is determined by its restriction to K and the values $v(t_1),\dots,v(t_r)$: the value of a polynomial in t and u is the least value of its monomials. We call such a valuation a **monomial valuation**. If $r = 0$, it is the **Gauss valuation** $v(\sum_\nu c_\nu u^\nu) = \min_\nu v(c_\nu)$.

We also use the following lemma, the proof of which is straightforward. Nevertheless, we include it for lack of a reference for the mixed characteristic case.

Lemma 1.3. *Let (K,v) be a henselian valued field of residue characteristic p that is closed under p-th roots. Denote $\Gamma_0 := \{0\}$ if K has characteristic p, and $\Gamma_0 := \mathbb{Z}v(p)$ otherwise. Then K has a subfield K_0 with $\Gamma_{K_0} = \Gamma_0$ and $k_{K_0} = k_K$.*

Proof. In equal characteristic this is [Kap42, Lemma 12], since K is perfect and henselian. In mixed characteristic the same argument works with $\mathbb{Z}v(p)$ in place of $\{0\}$, as follows. The restriction of v to $\mathbb{Q}$ is p-adic, so $\Gamma_{\mathbb{Q}} = \mathbb{Z}v(p)$. By Zorn's Lemma, let $K_0 \subseteq K$ be maximal among subfields of K with value group $\mathbb{Z}v(p)$, with respect to the valuation induced by v, and put $k_0 := k_{K_0}$. If $\alpha \in k_K$ were transcendental over k_0, then for any $a \in \mathcal{O}_K$ with $\overline{a} = \alpha$, the restriction of v to $K_0(a)$ would be the Gauss extension ([EP05, Corollary 2.2.2]), so

$\Gamma_{K_0(a)} = \mathbb{Z}v(p)$, contradicting the maximality of K_0. Hence k_K/k_0 is algebraic. Now suppose that $\alpha \in k_K \setminus k_0$ is either separably algebraic over k_0 or satisfies $\alpha^p \in k_0$. In the first case, Hensel's Lemma gives $a \in \mathcal{O}_K$ with $\overline{a} = \alpha$ that is a root of a monic lift to $\mathcal{O}_{K_0}[X]$ of the minimal polynomial of α over k_0. In the second case, we may take $a \in K$ with $a^p \in \mathcal{O}_{K_0}$ and $\overline{a^p} = \alpha^p$, using that K is closed under p-th roots. In either case we have $[K_0(a) : K_0] \leq [k_0(\alpha) : k_0] \leq [k_{K_0(a)} : k_0]$, so $\Gamma_{K_0(a)} = \mathbb{Z}v(p)$ by the Fundamental Equality, contradicting the maximality of K_0. Finally, every $\alpha \in k_K$ has some power α^{p^n} separably algebraic over k_0, and hence in k_0. Induction on n yields $\alpha \in k_0$, and therefore $k_K = k_0$. □

1.3. **Henselian fields: Artin–Schreier and Kummer extensions.**

Lemma 1.4. *Let (K, v) be henselian and of residue characteristic $p > 0$.*

(i) *If K is of characteristic p then $\mathfrak{m}_K \subseteq \wp(K)$.*
(ii) *If l is coprime to p then $1 + \mathfrak{m}_K \subseteq K^{\times l}$.*

Suppose now that K is of mixed characteristic and contains a primitive p-th root of unity.

(iii) *There is a $(p-1)$-th root ξ of $-p$ in K, and $v(\xi) = \frac{1}{p-1}v(p)$.*
(iv) *For $a, b \in 1 + \mathfrak{m}_K$, if $v(a-b) > \frac{p}{p-1}v(p)$ then $\frac{a}{b} \in K^{\times p}$.*
(v) *For $a, b \in K$ with $v(a) \geq \frac{1}{p-1}v(p)$ and $v(b^p) > v(p)$, we have $1 + a - pb \in (1 + a + b^p)K^{\times p}$.*

Proof. Both **(i)** and **(ii)** are standard exercises on Hensel's Lemma, and we omit the proofs. For **(iii)**: If $p = 2$ then take $\xi = -2$, so we may assume $p > 2$. Let $\zeta_p \in K$ be a primitive p-th root of unity, so that $\overline{\zeta_p} = 1$. Since $\prod_{i=1}^{p-1}(1 - \zeta_p^i) = p$ and $\overline{(\zeta_p^i - 1)/(\zeta_p - 1)} = i$, we have $-p(\zeta_p - 1)^{-(p-1)} \in 1 + \mathfrak{m}_K$. By **(ii)**, since $p-1$ is coprime to p, there is a $(p-1)$-th root x of $-p(\zeta_p - 1)^{-(p-1)}$ in K. Then $\xi := x(\zeta_p - 1)$ is the required $(p-1)$-th root of $-p$. Finally $v(x) = 0$, since x is a $(p-1)$-th root of a 1-unit. Therefore $v(\xi) = v(\zeta_p - 1) = \frac{1}{p-1}v(p)$.

For **(iv)**, let $a, b \in 1 + \mathfrak{m}_K$ be such that $v(a-b) > \frac{p}{p-1}v(p)$, and write $c := \frac{a}{b} - 1$, so that $v(c) = v(a-b)$ as $v(b) = 0$. It suffices to show $1 + c \in K^{\times p}$. Let $\xi \in K$ be as in **(iii)**, so $\xi^{p-1} = -p$. By substituting $X = \xi Y + 1$ into $X^p - (1 + c)$, and dividing by $\xi^p = -p\xi$, we obtain the polynomial

$$g(Y) := Y^p - Y + \sum_{i=2}^{p-1} \binom{p}{i} \xi^{i-p} Y^i - c\xi^{-p}.$$

Observe that $g \in K[Y]$. For $i \in \{2, \dots, p-1\}$ we have $v(\binom{p}{i}\xi^{i-p}) = (1 + \frac{i-p}{p-1})v(p) = \frac{i-1}{p-1}v(p) > 0$ and $v(c\xi^{-p}) = v(c) - \frac{p}{p-1}v(p) > 0$, the latter by our hypothesis on $v(a-b)$. Therefore $\overline{g}(Y) = Y^p - Y$, which has 0 as a simple root. By the henselianity of (K, v), there exists $y \in \mathcal{O}_K$ with $g(y) = 0$. Inverting the substitution, we have $x^p - (1 + c) = 0$ where $x = \xi y + 1$, i.e. $(\xi y + 1)^p = 1 + c$.

For **(v)**, we note that $v(1 + a - pb) = 0$, since $v(a) > 0$ and $v(pb) > v(p) > 0$. Since $v(b) > \frac{1}{p}v(p)$, for $2 \leq i \leq p-1$ we have $v\big(\binom{p}{i}b^i\big) \geq v(p) + 2v(b) > v(p) + v(b)$, and so there is a 1-unit $w \in 1 + \mathfrak{m}_K$ for which we have $(1 + b)^p = 1 + b^p + pbw$. Hence

$$\frac{1 + a + b^p}{1 + a - pb} - (1 + b)^p = \frac{b^p + pb}{1 + a - pb} - b^p - pbw = b^p\Big(\frac{1}{1 + a - pb} - 1\Big) + pb\Big(\frac{1}{1 + a - pb} - w\Big).$$

Both terms on the right-hand side of this equality have value $> \frac{p}{p-1}v(p)$: the first since $v(b^p) > v(p)$ and $v((1+a-pb)^{-1} - 1) = v(a - pb) \geq \frac{1}{p-1}v(p)$, the second since $v(pb) > v(p) + \frac{1}{p}v(p)$ and $v((1+a-pb)^{-1} - w) > 0$. As $(1 + b)^p \in K^{\times p}$, part **(iv)** applied to the 1-units $\frac{1+a+b^p}{1+a-pb}$ and $(1 + b)^p$ gives $\frac{1+a+b^p}{1+a-pb} \in K^{\times p}$, i.e. $1 + a - pb \in (1 + a + b^p)K^{\times p}$. □

Remark 1.5 (We need to talk about the substitution $X = \xi Y + 1$)**.** Let K be any field of characteristic zero, containing ξ such that $\xi^{p-1} = -p$, and let $a \in K$. Let $f = X^p - (1 + a)$. We let $h(Y) := \xi^{-p} f(\xi Y + 1)$, and readily we can compute (as in the proof of Lemma 1.4**(iii)**) that

$$h(Y) = Y^p - Y + \sum_{i=2}^{p-1} \binom{p}{i} \xi^{i-p} Y^i - c\xi^{-p}.$$

Clearly one of f and h is irreducible only if the other is, and if irreducible then they generate the same extension of K. If K admits a valuation v of residue characteristic p, then we see that h is equivalent to the Artin–Schreier polynomial $Y^p - Y - c\xi^{-p}$ modulo p.

Lemma 1.6. *Let (K,v) be a henselian valued field of positive characteristic with Γ_K p-divisible and k_K perfect. Let $a \in K$ be such that $v(a-\wp(x)) < 0$ for all $x \in K$, and let α be a root of $Z^p - Z - a$. Then $K(\alpha)/K$ is immediate of degree p.*

Proof. By assumption, $Z^p - Z - a$ has no root in K, therefore it is irreducible and $K(\alpha)/K$ is of degree p. Observe that $v(\alpha - x) < 0$ for all $x \in K$, since $\wp(\alpha - x) = a - \wp(x)$. This shows that $v(\alpha - K)$ has no maximum. To see this, suppose to the contrary that $\delta := v(\alpha - x_0)$ is the maximum, for some $x_0 \in K$. Since $\delta = v(\alpha - x_0) < 0$, we have $v(a - \wp(x_0)) = v(\wp(\alpha - x_0)) = p\delta$, so $p\delta \in \Gamma_K = p\Gamma_K$, and therefore $\delta \in \Gamma_K$. We choose $t \in K$ with $v(t) = -\delta$ and let $\chi := t(\alpha - x_0)$, so that $v(\chi) = 0$. Since $\chi^p - t^{p-1}\chi = t^p(a - \wp(x_0))$ and $v(t^{p-1}\chi) > 0$, we have that $\overline{\chi}^p = \overline{t^p(a - \wp(x_0))} \in k_K$. Since k_K is perfect, there is $u \in \mathcal{O}_K$ with $\overline{u}^p = \overline{\chi}^p$, and hence $\overline{u} = \overline{\chi}$. Then $v(\chi - u) > 0$, i.e. $v(\alpha - (x_0 + u/t)) > \delta$, which contradicts the maximality of δ.

Since K is henselian and $[K(\alpha) : K] = p$, the Fundamental Equality gives $p = efd$, where e, f, d are the ramification index, inertia degree, and defect of $K(\alpha)/K$. If $d = 1$, then $v(\alpha - K)$ has a largest element, by Observation 1.1**(ii)**, a contradiction. So $d = p$ and $e = f = 1$, i.e. $K(\alpha)/K$ is immediate. □

1.4. **Tame and separably tame fields.**

Definition 1.7. Let (K,v) be a henselian valued field. A finite extension L/K is **tame** if $e(L/K)$ is prime to p, k_L/k_K is separable, and $d(L/K) = 1$, i.e. $[L : K] = e(L/K)f(L/K)$. An algebraic extension is **tame** if all of its finite subextensions are tame [KPR86, §4]. The valued field (K,v) is [**separably**] **tame** if it is henselian and every finite [separable] extension of K is tame. An algebraic extension L/K is **purely wild** if Γ_L/Γ_K is a p-group and k_L/k_K is purely inseparable [KPR86, §4].

We write K^r for the absolute ramification field of (K,v), i.e. the ramification field of K^{sep}/K with respect to a fixed extension of v. It is the unique maximal tame extension of K [KPR86, Proposition 4.1].

Lemma 1.8 (see [Kuh16, Theorems 3.2 and 3.10])**.** *Let (K,v) be a henselian valued field of residue characteristic $p > 0$ with v nontrivial. Then (K,v) is separably tame if and only if it is separably algebraically maximal, Γ_K is p-divisible and k_K is perfect.*

Proof. This is standard. First suppose that (K,v) is separably tame. A finite separable immediate extension L/K is tame, so $[L : K] = e(L/K)f(L/K) = 1$, which shows that (K,v) is separably algebraically maximal. Let $\gamma \in \Gamma_K$ with $\gamma < 0$, and $c \in K$ with $v(c) = \gamma$. Put $g := X^p - X - c$ if $\mathrm{char}K = p$, and $g := X^p - c$ if $\mathrm{char}K = 0$. Then g is separable, and for every root α of g we have $v(\alpha) = \gamma/p$. Since $K(\alpha)/K$ is tame, p does not divide $e(K(\alpha)/K)$, so $\gamma/p \in \Gamma_K$. So Γ_K is p-divisible. Next we let $a \in \mathcal{O}_K^\times$. If $\mathrm{char}K = p$, choose $t \in K$ with $v(t) < 0$, and then write $g := X^p - X - at^p$; for a root α of g, we write $\beta := \alpha/t$. Then $v(\alpha) = v(t)$ and $\beta^p - t^{1-p}\beta = a$, where $v(t^{1-p}\beta) > 0$. If $\mathrm{char}K = 0$, we put $g := X^p - a$ and $\beta := \alpha$ for a root α of g. In either cases g is separable and $\overline{\beta}^p = \overline{a}$. As $K(\alpha)/K$ is tame, $k_{K(\alpha)}/k_K$ is separable, so the purely inseparable element $\overline{\beta}$ lies in k_K, which shows that k_K is perfect.

Conversely, suppose that (K,v) is separably algebraically maximal, Γ_K is p-divisible, and k_K is perfect. Let L/K be a finite separable purely wild extension. Then Γ_L/Γ_K is a p-group, hence trivial, as Γ_K is p-divisible and Γ_L is torsion-free. And k_L/k_K is purely inseparable, hence trivial, as k_K is perfect. So L/K is immediate, and $L = K$. Let W be a K-complement of K^r [KPR86, Theorem 2.1]. Finite subextensions of $(W \cap K^{\mathrm{sep}})/K$ are separable and linearly disjoint from K^r, hence purely wild [KPR86, Lemma 4.2], hence trivial. So W/K is purely inseparable, $\widetilde{K} = WK^r$ is purely inseparable over K^r, and $K^r = K^{\mathrm{sep}}$. As K^r is the maximal tame extension of K, it follows that (K,v) is separably tame. □

In characteristic p we write K^{perf} for the perfect hull of a field K, with the unique extension of the valuation.

Lemma 1.9 (cf. [KP16, Lemma 2.16])**.** *Let (K,v) be a separably tame field of characteristic $p > 0$. Then (K^{perf}, v) is tame. If v is nontrivial, then Γ_K is p-divisible, k_K is perfect, and K^{perf}/K is immediate.*

Proof. If v is trivial, then so is its extension to K^{perf}, and a trivially valued perfect field is tame. If, on the other hand, v is nontrivial, then Γ_K is p-divisible and k_K is perfect, by Lemma 1.8. Since $\Gamma_{K^{\mathrm{perf}}} = \frac{1}{p^\infty}\Gamma_K$ and $k_{K^{\mathrm{perf}}} = k_K^{\mathrm{perf}}$, the extension K^{perf}/K is immediate. Next we observe that K^{perf} is henselian, since it is algebraic over K. Let L'/K^{perf} be a finite extension, certainly separable, with primitive element c, and let n be such that the minimal polynomial of c over K^{perf} has coefficients in K^{1/p^n}. Then $b := c^{p^n}$ is separably algebraic over K, and $L' = K^{\mathrm{perf}}(b)$, since c is both separable and purely inseparable over $K^{\mathrm{perf}}(b)$. Since K is separably tame, $L := K(b)$ is a finite tame extension of K. It is clear that $L' = LK^{\mathrm{perf}} = L^{\mathrm{perf}}$, and $[L' : K^{\mathrm{perf}}] = [L : K]$ as L/K is separable. Moreover we have $\Gamma_{L'} = \frac{1}{p^\infty}\Gamma_L$ and $k_{L'} = k_L^{\mathrm{perf}}$. As $e(L/K)$ is prime to p and k_L/k_K is

separable, $e(L'/K^{\mathrm{perf}}) = e(L/K)$ and $f(L'/K^{\mathrm{perf}}) = f(L/K)$. So $[L' : K^{\mathrm{perf}}] = e(L'/K^{\mathrm{perf}})f(L'/K^{\mathrm{perf}})$, and therefore L'/K^{perf} is tame. □

Lemma 1.10 (Going down, cf. [Kuh16, Lemma 3.15])**.** *Let $(F/K, v)$ be an extension of valued fields with (F, v) separably tame, F/K regular, and k_F/k_K algebraic. Then (K, v) is separably tame, Γ_F/Γ_K is torsion-free, and k_F/k_K is purely inseparable. If v is nontrivial on K, then $k_F = k_K$.*

Proof. K is henselian, being relatively algebraically closed in F [EP05, Corollary 4.1.5]. By Hensel's Lemma every element of k_F separable over k_K is the residue of an element of F algebraic over K, hence of K. So k_F/k_K is purely inseparable. If v is trivial on K, then (K, v) is separably tame by definition, since every finite separable extension of K is trivially valued, and $\Gamma_F/\Gamma_K = \Gamma_F$ is torsion-free. So suppose that v is nontrivial on K.

Let L/K be a finite separable purely wild extension. By [KPR86, Lemma 4.2] and [EP05, Theorem 5.3.3(1)], $[L : K] = [LK^r : K^r] = p^n$ for some n. As F/K is regular, $[FL : F] = p^n$ and L is relatively algebraically closed in the henselian field FL. Since FL/F is tame of p-power degree, k_{FL}/k_F is separable of degree p^n. By Hensel's Lemma k_{FL}/k_L is purely inseparable, and k_L/k_K is purely inseparable as L/K is purely wild. Hence k_{FL}/k_K, and therefore k_{FL}/k_F, is purely inseparable. Being also separable of degree p^n, k_{FL}/k_F is trivial, so $n = 0$ and $L = K$. Now let W be a K-complement of K^r [KPR86, Theorem 2.1]. Finite subextensions of $(W \cap K^{\mathrm{sep}})/K$ are separable and linearly disjoint from K^r, hence purely wild [KPR86, Lemma 4.2], hence trivial. So W/K is purely inseparable, $\widetilde{K} = WK^r$ is purely inseparable over K^r, and $K^r = K^{\mathrm{sep}}$. As K^r is the maximal tame extension of K [KPR86, Proposition 4.1], (K, v) is separably tame.

Thus Γ_K is p-divisible and k_K is perfect by Lemma 1.8, so $k_F = k_K$. Let $\gamma \in \Gamma_F$ with $q\gamma \in \Gamma_K$ for a prime q. If $q = p$, then $\gamma \in \Gamma_K$ by p-divisibility. Otherwise, as $k_F = k_K$, there are $x \in F$ and $c \in K$ with $v(x) = \gamma$ and $x^q c^{-1} \in 1 + \mathfrak{m}_F$. By Lemma 1.4**(ii)**, $x^q c^{-1} = u^q$ with $u \in F$, so $y := xu^{-1}$ satisfies $y^q = c$, whence $y \in K$ and $\gamma = v(y) \in \Gamma_K$. □

1.5. **Kaplansky's theory of immediate extensions.** A **pseudo-Cauchy sequence** in a valued field (M, v) is a family $(c_\alpha)_{\alpha<\lambda}$ in M, indexed by a limit ordinal λ, such that $v(c_\beta - c_\alpha) < v(c_\gamma - c_\beta)$ for all $\alpha < \beta < \gamma$ (see [Ost35, p. 368] and [Kap42, §2]). Then $\gamma_\alpha := v(c_\beta - c_\alpha)$ does not depend on $\beta > \alpha$ by [Kap42, Lemma 2], and y is a **pseudo-limit** of the sequence if $v(y - c_\alpha) = \gamma_\alpha$ for all α. For $f \in M[X]$, the value $v(f(c_\alpha))$ is eventually constant or eventually strictly increasing, by [Kap42, Lemmas 1 and 5]. The sequence is of **transcendental type** if the first case holds for all f, and of **algebraic type** otherwise. We list three important results of Kaplansky, which we will use repeatedly:

- **(K1)** If $(L/M, v)$ is immediate, then every $z \in L \setminus M$ is a limit of a pseudo-Cauchy sequence in M without a limit in M [Kap42, Theorem 1].
- **(K2)** If z and z' are limits of a pseudo-Cauchy sequence in M of transcendental type, then they are transcendental over M, and $z \mapsto z'$ induces an isomorphism of valued fields $M(z) \to M(z')$ over M [Kap42, Theorem 2].
- **(K3)** Let a pseudo-Cauchy sequence $(c_\alpha)_\alpha$ in M of algebraic type have no limit in M, and let $q \in M[X]$ be of least degree such that $v(q(c_\alpha))$ is eventually strictly increasing. Then q is irreducible of degree at least 2, and for a root a of q, the valuation v extends to $M(a)$ so that $M(a)/M$ is immediate [Kap42, Theorem 3 and its proof].

Lemma 1.11 (cf. [KP16, Lemma 3.11])**.** *Let (M, v) be separably algebraically maximal. Then every pseudo-Cauchy sequence $(c_\alpha)_\alpha$ in M of algebraic type and without a limit in M is Cauchy, i.e. the γ_α are cofinal in Γ_M.*

Proof. Let $q \in M[X]$ be of least degree such that $v(q(c_\alpha))$ is eventually strictly increasing. By **(K3)**, adjoining a root of q gives a proper immediate extension of M. So q is not separable, as M is separably algebraically maximal. Hence $\mathrm{char} M = p$ and $q \in M[X^p]$, as q is irreducible. Suppose that some $\delta \in \Gamma_M$ exceeds all γ_α. By [Kap42, Lemma 8], there are $\beta \in \Gamma_M$ and $h \geq 1$ with $v(q(c_\alpha)) = \beta + h\gamma_\alpha < \beta + h\delta$ for all large α. The value $v(c_\alpha)$ is eventually constant, say equal to μ, since otherwise 0 would be a limit of $(c_\alpha)_\alpha$ [Kap42, Lemma 1]. Choose $\varepsilon \in M^\times$ with $v(\varepsilon) + \mu > \beta + h\delta$. Then $\tilde{q} := q + \varepsilon X$ has the same degree as q, and $v(\tilde{q}(c_\alpha)) = v(q(c_\alpha))$ for all large α. Moreover $\tilde{q}$ is separable, as its derivative is ε. So **(K3)** applied to $\tilde{q}$ gives a proper immediate separably algebraic extension of M, a contradiction. □

Remark 1.12. Lemma 1.11 is given in [KP16, Lemma 3.11] without the hypothesis that the sequence has no limit in M. This hypothesis is necessary: if $c \in M$ is a limit, then the sequence $v(c_\alpha - c) = \gamma_\alpha$ is strictly increasing, so $(c_\alpha)_\alpha$ is of algebraic type, witnessed by $X - c$, but it need not be Cauchy. This omission was pointed out by F.-V. Kuhlmann to P. Soto Moreno and the author, see also [SM25, Fact 2.27].

Corollary 1.13 (see [Kuh16, Corollary 3.12], [KP16, Lemma 2.16]). *Let (K,v) be a separably tame field of characteristic $p>0$ with v nontrivial. Then K is dense in K^{perf}.*

Proof. Let $a \in K^{\mathrm{perf}} \setminus K$, and let n be such that $a^{p^n} \in K$. As K^{perf}/K is immediate, by Lemma 1.9, **(K1)** makes a a limit of a pseudo-Cauchy sequence $(c_\alpha)_\alpha$ in K without a limit in K. It is of algebraic type, since $v(c_\alpha^{p^n} - a^{p^n}) = p^n\gamma_\alpha$ increases strictly. By Lemma 1.8, K is separably algebraically maximal, and therefore the values $v(a-c_\alpha)=\gamma_\alpha$ are cofinal in Γ_K, by Lemma 1.11. □

Lemma 1.14 (see [KP16, Lemma 3.11 and Proposition 3.10], [Kuh19a, Lemma 4.6]). *Let (M,v) be separably algebraically maximal, and let $(M(z)/M,v)$ be an immediate extension with z transcendental over M. Let $(c_\alpha)_\alpha$ be a pseudo-Cauchy sequence in M with limit z and without a limit in M, which exists by* **(K1)**.

- **(i)** *The sequence $(c_\alpha)_\alpha$ is of transcendental type.*
- **(ii)** *For every $f \in M[X]$ there is α such that $v(f(c)) = v(f(z))$ for all $c \in M$ with $v(z-c) \geq \gamma_\alpha$.*

Proof. The valuation v is nontrivial on M, as otherwise immediacy would force v to be trivial on $M(z)$, and then $M(z)=M$. For **(i)**, suppose that $v(f(c_\alpha))$ is eventually strictly increasing for some polynomial $f \in M[X]$. By Lemma 1.11, the $\gamma_\alpha = v(c_{\alpha+1}-c_\alpha)$ are cofinal in $\Gamma_M = \Gamma_{M(z)}$, and so the sequence $(c_\alpha)_\alpha$ converges to z. Consequently, the sequence $(f(c_\alpha))_\alpha$ converges to $f(z) \neq 0$, so $v(f(c_\alpha)) = v(f(z))$ for sufficiently large α, which is a contradiction.

For **(ii)**, suppose that f is nonconstant, and write $f(X) = \sum_i f_i(Y)(X-Y)^i$, with Hasse–Schmidt derivatives $f_i \in M[Y]$ [HS37]. So $f(c_\alpha) = f(z) + \sum_{i\geq 1} f_i(z)(c_\alpha - z)^i$, where $f(z)\neq 0$, and $f_i(z)\neq 0$ whenever $f_i \neq 0$. Put $m_f(\alpha) := \min\{v(f_i(z)) + i\gamma_\alpha \mid i \geq 1,\ f_i \neq 0\}$. This increases strictly with α, and for sufficiently large α the minimum is attained at a single i, since two of the functions $v(f_i(z)) + i\gamma_\alpha$ with different slopes agree for at most one α. If $m_f(\alpha) \leq v(f(z))$ for all α, then $v(f(c_\alpha)) = m_f(\alpha)$ for all sufficiently large α, which contradicts **(i)**. So $m_f(\alpha) > v(f(z))$ for some α. Therefore every $c \in M$ with $v(z-c) \geq \gamma_\alpha$ satisfies $v(f(c)-f(z)) \geq m_f(\alpha) > v(f(z))$, so $v(f(c)) = v(f(z))$. □

1.6. **p-bases.** We recall some basic facts on p-bases of a field E of characteristic $p>0$, mostly due to Mac Lane. A subset $A \subseteq E$ is **p-independent** if the monomials in A with exponents below p are linearly independent over E^p, and it is a **p-basis** of E if moreover $E = E^p(A)$.

- **(P1)** An extension F/E is separable if and only if some, equivalently every, p-basis of E remains p-independent in F. Then every p-independent subset of E remains p-independent in F. Every p-independent subset of F extends to a p-basis of F.
- **(P2)** If F/E is separably algebraic, then every p-basis B of E is a p-basis of F, as each $a \in F$ lies in $E(a^p) \subseteq F^p(B)$.
- **(P3)** If A is a p-basis of E and z is transcendental over E, then $A \cup \{z\}$ is a p-basis of $E(z)$.
- **(P4)** If F/E is separable and $E' \subseteq F$ is algebraic over E, then F/E' is separable.
- **(P5)** If (E,v) is a valued field and $B \subseteq \mathcal{O}_E^\times$ has p-independent residues, then B is p-independent. Indeed, dividing a relation $\sum_\mu c_\mu^p b^\mu = 0$ by a coefficient of least value and taking residues gives a relation between the residues.

An embedding of fields $\iota : E \to F$ is **separable** if $F/\iota(E)$ is separable. We say ι is **separated** if it is separable and $\mathrm{imp}(F/E) = 0$. For a separable extension F/E and a p-basis A of E, the **relative imperfection degree** $\mathrm{imp}(F/E)$ is the cardinality of any $C \subseteq F \setminus A$ such that $A \cup C$ is a p-basis of F, which does not depend on the choices, by the exchange property. We write $\mathrm{imp}(E) := \mathrm{imp}(E/\mathbb{F}_p)$ for the **imperfection degree** of a field E of characteristic p, i.e. the cardinality of a p-basis, so that $[E : E^p] = p^{\mathrm{imp}(E)}$ when $\mathrm{imp}(E)$ is finite. In characteristic 0, $\mathrm{imp}(E) := 0$. In characteristic 0 every extension and every embedding is separable and separated, and $\mathrm{imp}(F/E) := 0$.

2. Generalized Stability

Let $(F/K,v)$ be an extension of valued fields. An element $x \in F^\times$ is **valuation transcendental** over (K,v) if it is either "**value transcendental**", i.e. $v(x)$ has infinite order modulo Γ_K, or it is "**residue transcendental**", i.e. $v(x)=0$ and $\overline{x}$ is transcendental over k_K. We say $(F/K,v)$ is

- **inert** if $\Gamma_F = \Gamma_K$, the extension k_F/k_K is finite and separable, and $[F : K] = [k_F : k_K]$;
- **valuation regular** if Γ_F/Γ_K is torsion-free and k_F/k_K is regular;
- a **henselized function field** if (F,v) is the henselization of a finitely generated extension of K;
- **strongly inertially generated** if there are tuples t and u in F as in Lemma 1.2 such that $F/K(t,u)^{\mathrm{h}}$ is inert.

In this case the entries of t and u form a transcendence basis of F/K, the value group is $\Gamma_F = \Gamma_K \oplus \bigoplus_i \mathbb{Z}v(t_i)$, and the residue field $k_F/k_K(\overline{u})$ is finite and separable, by Lemma 1.2. In this section, we will only study the special case in which $\mathrm{trdeg}(F/K) = 1$: under this assumption, $(F/K, v)$ is strongly inertially generated if and only if there is $x \in F^\times$ that is valuation transcendental over (K, v), such that $F/K(x)^{\mathrm{h}}$ is inert.
The following straightforward lemma will be important to the proof of Theorem 1.

Lemma 2.1. *Let $(F/K, v)$ be a valuation regular extension of henselian valued fields, and let L/K be a finite extension (inside an algebraic closure of F, with any extension of v). Then $d(LF/F) \leq d(L/K)$.*

Proof. If $\gamma \in \Gamma_L \cap \Gamma_F$ then $\gamma + \Gamma_K$ is a torsion element of Γ_F/Γ_K, so $\gamma \in \Gamma_K$. Thus $\Gamma_L \cap \Gamma_F = \Gamma_K$ and $e(LF/F) \geq (\Gamma_L + \Gamma_F : \Gamma_F) = (\Gamma_L : \Gamma_K) = e(L/K)$. Since k_F/k_K is regular, it is linearly disjoint from the algebraic extension k_L/k_K, so $f(LF/F) \geq [k_F k_L : k_F] = [k_L : k_K] = f(L/K)$. As F and K are henselian and $[LF : F] \leq [L : K]$, the Fundamental Equality gives $d(LF/F) = [LF : F]/(e(LF/F)f(LF/F)) \leq [L : K]/(e(L/K)f(L/K)) = d(L/K)$. □

We introduce a standing hypothesis for the majority of this section:

(H) Let (K, v) be a henselian valued field of residue characteristic $p > 0$ that is closed under p-th roots, and if K has mixed characteristic then suppose it contains all p-th roots of unity. Let $(F/K, v)$ be a valuation regular henselized function field, without transcendence defect, of transcendence degree 1, and of rank 1.

Remark 2.2. From (H) it follows that

(i) k_K is perfect: since K is closed under p-th roots, so is its residue field. Since k_K is of characteristic p, this means that k_K is perfect.
(ii) Γ_K is p-divisible: since K is closed under p-th roots, its value group is closed under division by p.
(iii) Exactly one of the following holds
- (VT) The "value transcendental" case: $\Gamma_F = \Gamma_K \oplus \mathbb{Z}\gamma$ and $k_F = k_K$;
- (RT) The "residue transcendental" case: $\Gamma_F = \Gamma_K$ and k_F/k_K is a regular function field of transcendence degree 1.

The dichotomy between these two cases runs through the arguments in [Kuh10a].

Proof of **(iii)**. Since the extension is valuation regular, Γ_F/Γ_K is torsion-free and k_K is relatively algebraically closed in k_F. The Abhyankar Inequality, reinforced by the absence of transcendence defect, yields that $\mathrm{rrk}(\Gamma_F/\Gamma_K) + \mathrm{trdeg}(k_F/k_K) = 1$, which shows that $(\mathrm{rrk}(\Gamma_F/\Gamma_K), \mathrm{trdeg}(k_F/k_K))$ is either $(1, 0)$ or $(0, 1)$. The former is case (VT) and the latter is case (RT). In the case (VT), we take any $x \in F^\times$ with $v(x)$ of infinite order modulo Γ_K, so that the value group of $K(x)$ is $\Gamma_K \oplus \mathbb{Z}v(x)$ and the residue field is k_K. Since x is transcendental over K, $F/K(x)$ is a finite extension, and so both $(\Gamma_F : \Gamma_{K(x)})$ and $[k_F : k_{K(x)}]$ are finite, by the Fundamental Equality. Therefore Γ_F/Γ_K is a finitely generated torsion-free abelian group of rank 1, and $k_F = k_K$ (since k_F/k_K is regular, k_K is relatively algebraically closed in k_F, and $[k_F : k_K]$ is finite). Thus there exists $\gamma \in \Gamma_F$ such that $\Gamma_F = \Gamma_K \oplus \mathbb{Z}\gamma$, completing case (VT). In case (RT), we take any $y \in F$ with $\overline{y}$ transcendental over k_K. Then y is transcendental over K, and so $F/K(y)$ is a finite extension. By the Fundamental Equality, $k_F/k_{K(y)}$ is finite, which means that k_F/k_K is finitely generated (i.e. a function field) of transcendence degree 1, and it is regular since $(F/K, v)$ is valuation regular. □

Remark 2.3. The hypothesis (H) includes that v is of rank 1: this is used only four times in this whole section, twice each for two distinct uses, all of which occur in the verifcation of (SB-II) in the proof of Proposition 2.14. Twice it is used to argue that Γ_F is archimedean, and so therefore $n\alpha$ is cofinal in Γ_F whenever $\alpha > 0$, and twice it is used that the completion of a nontrivially valued subfield is henselian.

Definition 2.4. For a valued field (L, v) we define the associated **RV-structure** to be $\mathrm{RV}_L := L^\times/(1 + \mathfrak{m}_L)$: this is a multiplicative group. Let $\mathrm{rv} : L^\times \to \mathrm{RV}_L$ denote the quotient map. Since $1 + \mathfrak{m}_L$ is a subgroup of $\mathcal{O}_L^\times$, both the valuation and residue maps factor through rv, yielding the short exact sequence

$$1 \longrightarrow k_L^\times \longrightarrow \mathrm{RV}_L \xrightarrow{v} \Gamma_L \longrightarrow 0$$

If φ denotes the Frobenius map $x \mapsto x^p$, and Φ denotes the map it induces on each of $k_L^\times$, RV_L, and Γ_L (on the last of which it is multiplication by p), then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k_L^\times & \longrightarrow & \mathrm{RV}_L & \xrightarrow{v} & \Gamma_L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} & & \\ 1 & \longrightarrow & k_L^{\times p} & \longrightarrow & \mathrm{RV}_L^p & \longrightarrow & p\Gamma_L & \longrightarrow & 0 \end{array}$$

We denote $\mathrm{RV}_L^p := \{a^p \mid a \in \mathrm{RV}_L\}$, see for example [Kar23, §2.1]. The bottom row is exact as well: if $\mathrm{rv}(a)^p \in k_L^\times$ then $pv(a) = 0$, so $v(a) = 0$ since Γ_L is torsion-free, and hence $\mathrm{rv}(a)^p \in k_L^{\times p}$.

Lemma 2.5. *Let (L, v) be a valued field and let $x \in L^\times$.*

(i) $\mathrm{rv}(x) \in \mathrm{RV}_L^p$ *if and only if* $v(x) \in p\Gamma_L$ *and* $\overline{xy^{-p}} \in k_L^{\times p}$ *for some, equivalently every,* $y \in L^\times$ *with* $v(y^p) = v(x)$.

(ii) *If* $v(x) = 0$, *then* $\mathrm{rv}(x) \in \mathrm{RV}_L^p$ *if and only if* $\overline{x} \in k_L^{\times p}$.

(iii) *If k_L is perfect of characteristic p, then* $\mathrm{rv}(x) \in \mathrm{RV}_L^p$ *if and only if* $v(x) \in p\Gamma_L$.

Proof. First, we show that the choice of y in **(i)** does not matter: if $v(y^p) = v(y'^p)$ then $v(y) = v(y')$, since Γ_L is torsion-free, so $u := y'y^{-1} \in \mathcal{O}_L^\times$ and $\overline{xy'^{-p}} = \overline{xy^{-p}}\,\overline{u}^{-p}$. To prove **(i)**, we suppose first that $\mathrm{rv}(x) = \mathrm{rv}(z)^p$ for some $z \in L^\times$, i.e. $xz^{-p} \in 1 + \mathfrak{m}_L$. Then $v(x) = pv(z) \in p\Gamma_L$ and $\overline{xz^{-p}} = 1 \in k_L^{\times p}$, so the condition holds with $y := z$. Conversely, suppose that $v(x) = pv(y)$ and $\overline{xy^{-p}} = \alpha^p$ with $\alpha \in k_L^\times$, and choose $a \in \mathcal{O}_L^\times$ with $\overline{a} = \alpha$. Then $v(x(ay)^{-p}) = 0$ and $\overline{x(ay)^{-p}} = 1$, so $x(ay)^{-p} \in 1 + \mathfrak{m}_L$, i.e. $\mathrm{rv}(x) = \mathrm{rv}(ay)^p \in \mathrm{RV}_L^p$. Claim **(ii)** is just claim **(i)** with $y := 1$, and claim **(iii)** follows from **(i)**, since $k_L^{\times p} = k_L^\times$ when k_L is perfect of characteristic p. □

2.1. **Strong bases.** Sections of residue maps and cross-sections of valuations play huge roles in the structure theory of henselian valued fields, most notably in equal characteristic zero. These maps give appropriate lifts of the residue field and value groups, respectively, inside the value field itself, and go to some way towards literally splitting the valued field into two pieces: one each for the residue field and value group. Indeed, a section of the residue map is exactly a splitting of the map $K^\times \to k^\times$ and a cross-section is literally a splitting of $K^\times \to \Gamma$. In [Kuh10a, §4.2], Kuhlmann introduces the notion of a "Frobenius closed basis", which goes some way to unify these two types of lifts.

In the present note, we rename these "strong bases", to avoid conflict with the notion of a Frobenius closed basis of an extensions of fields (i.e. not *valued* fields), which are the subject of Appendix A. A small number of extra results on strong bases that are not used in any proof in the main body of this text are included in Appendix C. A key part of the definition is valuation independence, i.e. (SB-I), which appears in the literature in many forms: as "separated" (Baur [Bau82a, Bau82b], Delon [Del88]), as "orthogonal" (Bosch–Güntzer–Remmert [BGR84]), and as "valuation basis" (Cubides Kovacsics–Kuhlmann–Rzepka [CKR20]). We also lightly refine the theory. For a subset $\mathcal{B} \subseteq F$ we write $K\langle\mathcal{B}\rangle$ for the K-linear span of $\mathcal{B}$ in F.

Definition 2.6. A **strong basis** of $(F/K, v)$ is a set $\mathcal{B} \subseteq F^\times$ with $1 \in \mathcal{B}$ such that the following hold.

(SB-I) ("Valuation independent") $\mathcal{B}$ is valuation independent over (K, v), i.e.

$$v\Big(\sum_{b\in B} a_b b\Big) = \min\{v(a_b b) \mid b \in B\},$$

for all finite $B \subseteq \mathcal{B}$ and coefficients $a_b \in K$.

(SB-II) ("Span density") $K\langle\mathcal{B}\rangle$ is dense in (F, v).

(SB-III) ("Frobenius closed") $\mathcal{B}^p \subseteq \mathcal{B}$.

The set $B_0 = \{b \in B \mid v(a_b b) = \min_{c\in B} v(a_c c)\}$ is called the set of **dominant basis elements**, with respect to the sum $\sum_{b\in B} a_b b$.

Remark 2.7. Span density, i.e. (SB-II), is what is needed whenever an approximation to high precision is then expanded as $\sum_b a_b b$, as in for example Lemma 2.21, where the precision must exceed $\frac{p}{p-1}v(p)$, and in the Artin–Schreier case of Proposition 2.22, where elements of negative value must be approximated modulo $\mathfrak{m}_F$. There are at least two natural alternative axioms around the notion of density:

(i) "Density of the ring $K[\mathcal{B}]$." Since $K\langle\mathcal{B}\rangle \subseteq K[\mathcal{B}]$, this is implied by (SB-II), and the two agree when $K\langle\mathcal{B}\rangle$ is a subring of F, as in case (VT) (Proposition 2.14**(i)**). In general they need not agree: a product bb' of elements of $\mathcal{B}$ lies in $K[\mathcal{B}]$, but need not lie in the closure of $K\langle\mathcal{B}\rangle$.

(ii) "First-order density": for every $x \in F^\times$ there is $s \in K\langle\mathcal{B}\rangle$ with $v(x-s) > v(x)$, so that $\mathrm{rv}(s) = \mathrm{rv}(x)$. Together with (SB-I), this says that the elements $\mathrm{rv}(b)$, for $b \in \mathcal{B}$, form a basis of the graded ring

$$\mathrm{gr}_v(F) := \bigoplus_{\gamma\in\Gamma_F} \{x \mid v(x) \geq \gamma\}/\{x \mid v(x) > \gamma\}$$

over $\mathrm{gr}_v(K)$. For this to make sense, we identify $\mathrm{RV}_F \cup \{0\}$ with its image in $\mathrm{gr}_v(F)$ via $\mathrm{rv}(x) \mapsto x + \{y \mid v(y) > v(x)\}$. It follows from (SB-II) (Lemma 2.8**(i)**), and it is all that is needed for arguments that only see rv. However, the converse fails in general: iterative use of first-order density yields approximations that increase in value, but may stay below a fixed element of the value group. This happens in the example of Remark C.4, where $\Gamma_F = \mathbb{Z}\times\mathbb{Z}$ has rank 2. If $\Gamma_F \cong \mathbb{Z}$, i.e. it is discrete and of rank 1, then unbounded sets of values are cofinal, and so first-order density implies (SB-II).

First, some straightforward consequences of the definition.

Lemma 2.8 (Consequences of span density)**.** *Assume (H) and let $\mathcal{B}$ be a strong basis of $(F/K, v)$.*

(i) *For every $x \in F^\times$ there is $s \in K\langle\mathcal{B}\rangle$ with $v(x-s) > v(x)$, so in particular $\mathrm{rv}(s) = \mathrm{rv}(x)$.*
(ii) *Every coset of Γ_K in Γ_F contains $v(b)$ for some $b \in \mathcal{B}$.*
(iii) *For every $\varepsilon \in k_F^\times$ there is $s \in K\langle\mathcal{B}\rangle$ with $v(s) = 0$ and $\overline{s} = \varepsilon$.*

Proof. For **(i)**: by (SB-II) there is $s \in K\langle\mathcal{B}\rangle$ with $v(x-s) > v(x)$, and then $v(s) = v(x)$ and $\mathrm{rv}(s) = \mathrm{rv}(x)$. For **(ii)**: let $\delta \in \Gamma_F$ and choose $t \in F^\times$ with $v(t) = \delta$. By **(i)** there is $s \in K\langle\mathcal{B}\rangle$ with $v(t-s) > \delta$, so that $v(s) = \delta$. Writing $s = \sum_{b\in B} a_b b$ with $a_b \in K^\times$, property (SB-I) yields $b \in B$ with $v(a_b b) = \delta$, whence $v(b) \in \delta + \Gamma_K$. For **(iii)**: choose $u \in \mathcal{O}_F^\times$ with $\overline{u} = \varepsilon$ and, by **(i)**, there exists $s \in K\langle\mathcal{B}\rangle$ with $v(s-u) > 0$. Then $v(s) = 0$ and $\overline{s} = \overline{u} = \varepsilon$. □

Part **(ii)** of the following lemma will not be used in the rest of these notes. Nevertheless, we include it for completeness, since in particular it describes the residues of a strong basis in case (RT).

Lemma 2.9 (Consequences of valuation independence)**.** *Assume (H), and let $\mathcal{B} \subseteq F^\times$ satisfy (SB-I).*

(i) *Suppose case (VT) holds. Then distinct elements of $\mathcal{B}$ have values in distinct cosets of Γ_K. Consequently, for any finite $B \subseteq \mathcal{B}$ and coefficients $a_b \in K^\times$, the values $v(a_b b)$, for $b \in B$, are pairwise distinct, and B_0 is a singleton.*
(ii) *Suppose case (RT) holds. For $b \in \mathcal{B}$ choose $t_b \in K^\times$ with $v(t_b b) = 0$, taking $t_1 = 1$, and put $\varepsilon_b := \overline{t_b b} \in k_F^\times$. Then the family $(\varepsilon_b)_{b\in\mathcal{B}}$ is k_K-linearly independent. If moreover $\mathcal{B}$ is a strong basis, then $\{\varepsilon_b \mid b \in \mathcal{B}\}$ is a k_K-basis of k_F.*

Proof. First suppose (VT). Suppose $b \neq b'$ in $\mathcal{B}$ with $v(b) - v(b') \in \Gamma_K$, and choose $a \in K^\times$ with $v(a) = v(b) - v(b')$, so that $v(ab') = v(b)$. In case (VT) we have $k_F = k_K$, so there is $\lambda \in \mathcal{O}_K^\times$ with $\overline{\lambda} = \overline{b(ab')^{-1}}$, whence $v(b - \lambda ab') > v(b) = \min\{v(b), v(\lambda ab')\}$. This contradicts (SB-I) applied to $\{b, b'\}$ with coefficients 1 and $-\lambda a$.

Next suppose (RT). Such t_b exist since $\Gamma_F = \Gamma_K$ in case (RT). Notice that changing t_b multiplies ε_b by an element of $k_K^\times$, and neither of the conclusions of the lemma depends on this choice. Let $B \subseteq \mathcal{B}$ be finite, and let $\alpha_b \in k_K$ satisfy $\sum_{b\in B} \alpha_b \varepsilon_b = 0$. We argue that all α_b are zero. For each $b \in B$ choose $a_b \in \mathcal{O}_K$ with $\overline{a_b} = \alpha_b$, taking $a_b = 0$ when $\alpha_b = 0$. Then $\overline{\sum_{b\in B} a_b t_b b} = \sum_{b\in B} \alpha_b \varepsilon_b = 0$, so $v(\sum_{b\in B} a_b t_b b) > 0$, and by (SB-I) $v(a_b) = v(a_b t_b b) > 0$ for every $b \in B$, i.e. $\alpha_b = 0$.

Finally, suppose that $\mathcal{B}$ is a strong basis, and let $\varepsilon \in k_F^\times$. By Lemma 2.8**(iii)** there is $y \in K\langle\mathcal{B}\rangle$ with $v(y) = 0$ and $\overline{y} = \varepsilon$. Write $y = \sum_{d\in D} c_d d$ with $D \subseteq \mathcal{B}$ finite and $c_d \in K^\times$, and let $D_0 := \{d \in D \mid v(c_d d) = 0\}$. By (SB-I), $v(c_d d) \geq v(y) = 0$ for all $d \in D$, so

$$\varepsilon = \overline{y} = \sum_{d\in D_0} \overline{c_d d} = \sum_{d\in D_0} \overline{c_d t_d^{-1}}\, \varepsilon_d,$$

with $\overline{c_d t_d^{-1}} \in k_K$. Hence the ε_b span k_F over k_K. □

Strong bases automatically enjoy the following property of "multiplicative rigidity".

Lemma 2.10. *Assume (H) and let $\mathcal{B}$ be a strong basis. The following holds.*

(SB-IV) *("Multiplicative rigidity") If $B \subseteq \mathcal{B}$ is finite and nonempty, and $a_b \in K^\times$ are coefficients, then*

$$\mathrm{rv}\Big(\sum_{b\in B} a_b b\Big) \in \mathrm{RV}_F^p \implies B_0 \subseteq \mathcal{B}^p,$$

where $B_0 = \{b \in B \mid v(a_b b) = \min_{c\in B} v(a_c c)\}$.

Proof. Let $B \subseteq \mathcal{B}$ be finite and nonempty, let $a_b \in K^\times$ for $b \in B$, put $a := \sum_{b\in B} a_b b$, and suppose that $\mathrm{rv}(a) \in \mathrm{RV}_F^p$. By (SB-I), $v(a) = v(a_b b)$ for every $b \in B_0$, and by Lemma 2.5**(i)**, $v(a) \in p\Gamma_F$. By Remark 2.2**(iii)**, exactly one of the cases (VT) and (RT) holds.

First suppose that case (VT) holds. By Lemma 2.9**(i)**, $B_0 = \{b_0\}$ is a singleton. Since Γ_K is p-divisible (Remark 2.2**(ii)**), we have $v(a_{b_0}) \in p\Gamma_K$, and so $v(b_0) = v(a) - v(a_{b_0}) \in p\Gamma_F$. Write $v(b_0) = p\delta$ with $\delta \in \Gamma_F$. By Lemma 2.8**(ii)** there is $b' \in \mathcal{B}$ with $v(b') \in \delta + \Gamma_K$. Then $b'^p \in \mathcal{B}$ by (SB-III), and $v(b'^p) \in p\delta + p\Gamma_K \subseteq v(b_0) + \Gamma_K$. By Lemma 2.9**(i)**, $b'^p = b_0$. Hence $B_0 \subseteq \mathcal{B}^p$.

Next suppose that case (RT) holds, and choose $t_b \in K^\times$ and $\varepsilon_b = \overline{t_b b}$, for $b \in \mathcal{B}$, as in Lemma 2.9**(ii)**. Since $\Gamma_F = \Gamma_K$ is p-divisible, there is $y \in K^\times$ with $v(y^p) = v(a)$, and then $\overline{ay^{-p}} \in k_F^{\times p}$ by Lemma 2.5**(i)**. For $b \in B$ we have $v(a_b y^{-p} t_b^{-1}) = v(a_b b) - v(a) \geq 0$, by (SB-I). So $\lambda_b := \overline{a_b y^{-p} t_b^{-1}} \in k_K$, and $\lambda_b \neq 0$ if and only if $b \in B_0$. Since $ay^{-p} = \sum_{b\in B}(a_b y^{-p} t_b^{-1})(t_b b)$ is a sum of elements of $\mathcal{O}_F$,

$$\overline{ay^{-p}} = \sum_{b\in B} \lambda_b \varepsilon_b.$$

On the other hand, let $d \in \mathcal{B}$. Then $d^p \in \mathcal{B}$ by (SB-III), and $v(t_d^p d^p) = 0 = v(t_{d^p} d^p)$, so $u_d := t_d^p t_{d^p}^{-1} \in \mathcal{O}_K^\times$ and $\varepsilon_d^p = \overline{u_d}\varepsilon_{d^p}$. By Lemma 2.9**(ii)**, every element of k_F has the form $\sum_{d\in D} \mu_d \varepsilon_d$, with $D \subseteq \mathcal{B}$ finite and $\mu_d \in k_K$. Since k_F has characteristic p, its p-th power is $\sum_{d\in D} \mu_d^p \overline{u_d}\varepsilon_{d^p}$. Thus k_F^p lies in the k_K-linear span of $\{\varepsilon_c \mid c \in \mathcal{B}^p\}$. In particular $\sum_{b\in B} \lambda_b \varepsilon_b = \sum_{c\in C} \nu_c \varepsilon_c$ for some finite $C \subseteq \mathcal{B}^p$ and $\nu_c \in k_K$. The family $(\varepsilon_b)_{b\in\mathcal{B}}$ is k_K-linearly independent, by Lemma 2.9**(ii)**. Comparing coefficients, every $b \in B$ with $\lambda_b \neq 0$ lies in C. Hence $B_0 \subseteq C \subseteq \mathcal{B}^p$. □

Remark 2.11. Observe that, given (SB-I), the property (SB-IV) depends only on the sub-sum $\sum_{b\in B_0} a_b b$ since in turn rv depends only on elements of equal-minimal value.

Remark 2.12. Assume (H). Let $\mathcal{B}$ be a strong basis of $(F/K, v)$, let $B \subseteq \mathcal{B}$ be finite and nonempty, and let $a_b \in K^\times$ for $b \in B$. Denote $a = \sum_{b\in B} a_b b$ and let B_0 be the set of dominant basis elements. Suppose that $B_0 \not\subseteq \mathcal{B}^p$. By (SB-IV) we have $\mathrm{rv}(a) \notin \mathrm{RV}_F^p$. Moreover

- in case (VT), $k_F = k_K$ is perfect, so Lemma 2.5**(iii)** gives $v(a) \notin p\Gamma_F$;
- in case (RT), if $v(a) = 0$ then Lemma 2.5**(ii)** gives $\overline{a} \notin k_F^{\times p}$.

The following lemma is a simple form of full "Additive rigidity", proved in Lemma C.1. The latter is not used in the rest of this paper. See also Remark C.2.

Lemma 2.13 ("Additive rigidity" – simple form)**.** *Let $(F/K, v)$ be an extension of valued fields of characteristic p with a strong basis $\mathcal{B}$. Let $a \in K$ with $v(a - \wp(x)) < 0$ for all $x \in K$. Then $v(a - \wp(y)) < 0$ for all $y \in F$. I.e. $K \cap (\wp(F) + \mathcal{O}_F) = \wp(K) + \mathcal{O}_K$. In particular $a \notin \wp(F)$, i.e. $Z^p - Z - a$ is irreducible over F.*

Proof. The inclusion $K \cap (\wp(F) + \mathcal{O}_F) \supseteq \wp(K) + \mathcal{O}_K$ is trivial. Conversely, let $y \in F$ and $z \in \mathcal{O}_F$, and suppose that $x := \wp(y) + z$ is an element of K. By (SB-II), there exists $w \in K\langle\mathcal{B}\rangle$ with $v(y - w) > 0$. Then $\wp(y) - \wp(w) = \wp(y - w)$ has value $v(y - w) > 0$, so $x - \wp(w) \in \mathcal{O}_F$. We may, therefore, write $w = w_1 + \sum_{b\in B} w_b b$, for a finite support $B \subseteq \mathcal{B} \setminus \{1\}$, and $w_b \in K$. Observe that $x - \wp(w) = x - \wp(w_1) - \sum_{b\in B}(w_b^p b^p - w_b b)$. By (SB-III), each b^p lies in $\mathcal{B}$, and $b^p \neq 1$ for $b \neq 1$ since Frobenius is injective. So this is again an element of the form

$$x - \wp(w_1) + \sum_{b\in B'} w'_b b,$$

for $B' \subseteq \mathcal{B} \setminus \{1\}$. By (SB-I), $v(x - \wp(w_1)) \geq v(x - \wp(w)) \geq 0$. And so $x = \wp(w_1) + (x - \wp(w_1)) \in \wp(K) + \mathcal{O}_K$. Finally, let $a \in K$ with $v(a - \wp(x)) < 0$ for all $x \in K$, i.e. $a \notin \wp(K) + \mathcal{O}_K$. By what we have shown, $a \notin \wp(F) + \mathcal{O}_F$, i.e. $v(a - \wp(y)) < 0$ for all $y \in F$. In particular $a \notin \wp(F)$, so $Z^p - Z - a$ has no root in F, and an Artin–Schreier polynomial without a root is irreducible. □

The following proposition shows that strong bases exist in strongly inertially generated function fields. Its proof is the only one in the body of this paper to use the results of Appendix A on Frobenius closed bases of algebraic function fields.

Proposition 2.14 (Strong bases exist)**.** *Continue to assume (H). Suppose that $(F/K, v)$ is strongly inertially generated. The extension $(F/K, v)$ admits a strong basis $\mathcal{B}$. Moreover $\mathcal{B}$ may be chosen so that:*

(i) *In case (VT), $\mathcal{B} = \{t^n \mid n \in \mathbb{Z}\}$ for some $t \in F^\times$ such that $\Gamma_F = \Gamma_K \oplus \mathbb{Z}v(t)$.*
(ii) *In case (RT), $v(b) = 0$ for all $b \in \mathcal{B}$.*

Proof. In case (VT), let $t \in F^\times$ be valuation transcendental over (K,v) with $F/K(t)^{\mathrm{h}}$ inert. By Remark 2.2**(iii)** we have $k_F = k_K$. Hence t is not residue transcendental, so it is value transcendental, and therefore $k_{K(t)} = k_K$ and $\Gamma_{K(t)} = \Gamma_K \oplus \mathbb{Z}v(t)$. Since $(F/K(t)^{\mathrm{h}}, v)$ is inert, we have $[F : K(t)^{\mathrm{h}}] = [k_F : k_{K(t)}] = 1$, therefore $F = K(t)^{\mathrm{h}}$, and $\Gamma_F = \Gamma_K \oplus \mathbb{Z}v(t)$ since henselization is immediate. Define $\chi : \mathbb{Z}v(t) \to F^\times$ by $\chi(nv(t)) := t^n$. Then $\mathcal{B} := \chi(\mathbb{Z}v(t)) = \{t^n \mid n \in \mathbb{Z}\}$ is a strong basis:

First observe that $1 = t^0 \in \mathcal{B}$. For (SB-I): let $B \subseteq \mathcal{B}$ be finite and let $a_b \in K$ be not all zero. After discarding those b with $a_b = 0$, we may assume all $a_b \in K^\times$. If $b = t^n$ and $b' = t^{n'}$ are distinct elements of B, then $v(a_b b) - v(a_{b'} b') = (v(a_b) - v(a_{b'})) + (n - n')v(t) \neq 0$, since $(n-n')v(t) \notin \Gamma_K$. Thus the values $v(a_b b)$ are pairwise distinct, so the minimum is attained exactly once, and by the Ultrametric Inequality $v(\sum_{b\in B} a_b b) = \min\{v(a_b b) \mid b \in B\}$. For (SB-II): since v is of rank 1, completions are henselian, and so $K(t)^{\mathrm{h}}$ embeds into the completion of $K(t)$, over $K(t)$. Thus $K(t)$ is dense in $K(t)^{\mathrm{h}}$. Moreover $K\langle\mathcal{B}\rangle = K[t,t^{-1}]$ is dense in $K(t)$ by a standard argument using the geometric series, and the rank 1 hypothesis, as follows. Indeed, let $f = \sum_i a_i t^i \in K[t,t^{-1}]$ be any nonzero polynomial in t, t^{-1} over K, and let m be the index at which $\min_i v(a_i t^i)$ is attained, which is unique by (SB-I). Then $f = a_m t^m (1-y)$ where $y := -\sum_{i\neq m} a_m^{-1} a_i t^{i-m} \in K[t,t^{-1}]$ satisfies $v(y) > 0$. Since v is of rank 1, the value group Γ_F is archimedean, so $v(y^j) = jv(y)$ is cofinal in Γ_F, as $j \to \infty$. As $f^{-1} = a_m^{-1} t^{-m} \sum_{j\geq 0} y^j$ (a series that converges in the completion of F), the partial sums $a_m^{-1} t^{-m} \sum_{j=0}^{J} y^j$, which lie in $K[t,t^{-1}]$, converge to f^{-1}. Since $K[t,t^{-1}]$ is closed under multiplication, every element g/f of $K(t)$ is likewise approximated by elements of $K[t,t^{-1}]$. For (SB-III), it is clear that $\mathcal{B}^p = \{t^{pn} \mid n \in \mathbb{Z}\} \subseteq \mathcal{B}$.

In case (RT), we have $\Gamma_F = \Gamma_K$, and k_F/k_K is a regular function field of transcendence degree 1 with k_K perfect, by Remark 2.2**(i,iii)**. Let $u \in F^\times$ be valuation transcendental over (K,v) with $F/K(u)^{\mathrm{h}}$ inert. Since $\Gamma_F = \Gamma_K$, the element u is residue transcendental, i.e. $v(u) = 0$ and $\overline{u}$ is transcendental over k_K. As $F/K(u)^{\mathrm{h}}$ is inert, $k_F/k_K(\overline{u})$ is finite and separable. Next we choose subfields $K_0 \subseteq K$ and $F_0 \subseteq F$ whose value groups are discrete, but which have the same residue fields as K and F. Let $K_0 \subseteq K$ be as in Lemma 1.3, so $k_{K_0} = k_K$ and $\Gamma_{K_0} = \Gamma_0$ is discrete. Since $\overline{u}$ is transcendental over k_{K_0}, by [EP05, Corollary 2.2.2], $\Gamma_{K_0(u)} = \Gamma_0$ and $k_{K_0(u)} = k_K(\overline{u})$. Let η be a primitive element of $k_F/k_K(\overline{u})$, and let $g \in \mathcal{O}_{K_0(u)}[Y]$ be a monic polynomial such that $\overline{g}$ is the minimal polynomial of η over $k_K(\overline{u})$. Since η is a simple root of $\overline{g}$ and (F,v) is henselian, there is $u' \in \mathcal{O}_F$ with $g(u') = 0$ and $\overline{u'} = \eta$. Put $F_0 := K_0(u,u') \subseteq F$. Then $[F_0 : K_0(u)] \leq \deg(g) = [k_F : k_K(\overline{u})] \leq [k_{F_0} : k_{K_0(u)}]$, so by the Fundamental Equality $\Gamma_{F_0} = \Gamma_{K_0(u)} = \Gamma_0$ and $k_{F_0} = k_F$. Moreover $K(u)^{\mathrm{h}}(u') \subseteq F$ and $[K(u)^{\mathrm{h}}(u') : K(u)^{\mathrm{h}}] \geq [k_F : k_K(\overline{u})]$. Since $F/K(u)^{(\mathrm{H})}$ is inert, we have $[k_F : k_K(\overline{u})] = [F : K(u)^{\mathrm{h}}]$, and hence $F = K(u)^{\mathrm{h}}(u')$. It follows that F is the henselization of the compositum $KF_0 = K(u,u')$: indeed $K(u,u')^{\mathrm{h}} \subseteq F$, as F is henselian, and $K(u,u')^{\mathrm{h}}$ contains $K(u)^{\mathrm{h}}$ and u'. In equal characteristic, K_0 and F_0 are trivially valued, and the residue map restricts to isomorphisms $K_0 \to k_K$ and $F_0 \to k_F$, whereas in mixed characteristic we have $\Gamma_{F_0} = \mathbb{Z}v(p)$.

By Theorem A.7 and Lemma A.3, k_F has a k_K-basis $\mathcal{C}$ containing 1 which is freely Frobenius generated by some $E \subseteq k_F$. For each $\varepsilon \in E$ choose $b_\varepsilon \in \mathcal{O}_{F_0}$ with $\overline{b_\varepsilon} = \varepsilon$, and write

$$\mathcal{B} := \{1\} \cup \{b_\varepsilon^{p^n} \mid \varepsilon \in E,\ n \in \mathbb{N}\} \subseteq F_0.$$

We have $v(b) = 0$ for all $b \in \mathcal{B}$, which is assertion **(ii)** of the present lemma. The residue map sends $\mathcal{B}$ bijectively onto $\mathcal{C}$, commuting with p-th powers, since the elements ε^{p^n} are pairwise distinct by Lemma A.3. Then (SB-III) is immediate from the choice of $\mathcal{B}$. For (SB-I), let $B \subseteq \mathcal{B}$ be finite and let $a_b \in K$ be not all zero. After discarding those b with $a_b = 0$, we may assume that $a_b \in K^\times$ for all $b \in B$. Since $\Gamma_F = \Gamma_K$ there is $d \in K^\times$ with $v(d) = \delta := \min_{b\in B} v(a_b)$, and $v(a_b b) = v(a_b)$ as $v(b) = 0$. Then $\overline{d^{-1}\sum_{b\in B} a_b b} = \sum_{b\in B} \overline{a_b d^{-1}}\,\overline{b}$, with $\overline{a_b d^{-1}} \in k_K$ and at least one of them nonzero. Since $\overline{B} \subseteq \mathcal{C}$ is k_K-linearly independent, this residue is nonzero, so $v(\sum_{b\in B} a_b b) = \delta = \min_{b\in B} v(a_b b)$.

For (SB-II), we argue in three steps. Firstly, we show that $K_0\langle\mathcal{B}\rangle$ is dense in F_0. Let $y \in F_0^\times$. Since $\Gamma_{F_0} = \Gamma_{K_0}$ there is $d \in K_0^\times$ with $v(d) = v(y)$, and since $\overline{\mathcal{B}} = \mathcal{C}$ spans $k_{F_0} = k_F$ over $k_{K_0} = k_K$, there is $s \in K_0\langle\mathcal{B}\rangle$ with $v(s) = 0$ and $\overline{s} = \overline{yd^{-1}}$ (lift the coefficients of $\overline{yd^{-1}}$ with respect to $\mathcal{C}$ to $\mathcal{O}_{K_0}$, and use (SB-I)). Thus $ds \in K_0\langle\mathcal{B}\rangle$ and $v(y - ds) > v(y)$. Iterating this process, we obtain elements of $K_0\langle\mathcal{B}\rangle$ whose distances to y have strictly increasing values in Γ_{F_0}, unless y itself is reached after finitely many steps. As Γ_{F_0} is discrete and v is of rank 1, these values are cofinal in Γ_F.

Secondly, we show that $K\langle\mathcal{B}\rangle$ is dense in the ring $K[F_0]$ generated by K and F_0. Indeed, an element of $K[F_0]$ is a finite sum $\sum_i k_i f_i$ with $k_i \in K$ and $f_i \in F_0$. Approximating each f_i by some $g_i \in K_0\langle\mathcal{B}\rangle$ gives $\sum_i k_i g_i \in K\langle\mathcal{B}\rangle$ with $v(\sum_i k_i(f_i - g_i)) \geq \min_i(v(k_i) + v(f_i - g_i))$.

Thirdly, we show that $K[F_0]$ is dense in F. Let $r \in K[F_0] \setminus \{0\}$. As $v(r) \in \Gamma_F = \Gamma_K$ there is $s_1 \in K^\times$ with $v(rs_1) = 0$, and as $k_{F_0} = k_F$ there is $s_2 \in F_0$ with $v(s_2) = 0$ and $\overline{s_2} = \overline{rs_1}^{-1}$. Putting $s := s_1 s_2 \in K[F_0]$ we have

$z := 1 - rs \in \mathfrak{m}_F$, so $r^{-1} = s\sum_{j\geq 0} z^j$, and the partial sums lie in $K[F_0]$ and converge to r^{-1}, since $v(z^j) = jv(z)$ is cofinal in Γ_F, v being of rank 1. Since $K[F_0]$ is closed under multiplication, it is dense in its field of fractions KF_0. Finally, since v is of rank 1, KF_0 is dense in its henselization $(KF_0)^{\mathrm{h}} = F$, as in case (VT).

These three steps together show that $K\langle\mathcal{B}\rangle$ is dense in (F, v), which is exactly the statement (SB-II). In equal characteristic, since the residue map restricts to an isomorphism $F_0 \to k_F$, we even have $K_0\langle\mathcal{B}\rangle = F_0$, so $K\langle\mathcal{B}\rangle = K[F_0]$ is a subring of F (cf. Remark 2.7). □

Remark 2.15. In Proposition 2.14, the hypothesis that $(F/K, v)$ is strongly inertially generated cannot be omitted without suitable replacement. The conclusion of the lemma may be false when $(F/K(c)^{\mathrm{h}}, v)$ has nontrivial defect for an element $c \in F$ that is valuation transcendental. In this case $\{c^n \mid n \in \mathbb{Z}\}$ fails to satisfy (SB-II), for this particular element c.

Lemma 2.16 (Immediate extensions of the base)**.** *Let $\mathcal{B}$ be a strong basis of $(F/K, v)$, and let L/K be a finite immediate extension with $[FL : F] = [L : K]$. Then $(FL/F, v)$ is immediate.*

Proof. First let $s = \sum_{b\in B} \eta_b b$ with $B \subseteq \mathcal{B}$ finite and $\eta_b \in L^\times$. Since L/K is immediate, there are $\kappa_b \in K$ with $v(\eta_b - \kappa_b) > v(\eta_b)$, so $v(\kappa_b) = v(\eta_b)$. Put $z := \sum_{b\in B} \kappa_b b \in K\langle\mathcal{B}\rangle$. By (SB-I), $v(z) = \min_{b\in B} v(\kappa_b b) = \min_{b\in B} v(\eta_b b)$, while $v(s - z) \geq \min_{b\in B} v((\eta_b - \kappa_b)b) > \min_{b\in B} v(\eta_b b)$. Hence $v(s) = \min_{b\in B} v(\eta_b b)$ and $v(s - z) > v(s)$.

Next, we show that $L\langle\mathcal{B}\rangle$ is dense in FL. Let $\varepsilon_1, \dots, \varepsilon_n$ be a K-basis of L. It is an F-basis of FL, as $[FL : F] = [L : K]$. Let $y = \sum_j f_j\varepsilon_j$ with $f_j \in F$, and let $\gamma \in \Gamma_{FL}$. Since Γ_{FL}/Γ_F is torsion, there is $\delta \in \Gamma_F$ with $\delta \geq \gamma - \min_j v(\varepsilon_j)$. By (SB-II) there are $s_j \in K\langle\mathcal{B}\rangle$ with $v(f_j - s_j) > \delta$. Thus $\sum_j s_j\varepsilon_j \in L\langle\mathcal{B}\rangle$ and $v(y - \sum_j s_j\varepsilon_j) > \gamma$.

Finally let $y \in (FL)^\times$. By density there is $s \in L\langle\mathcal{B}\rangle$ with $v(y - s) > v(y)$, so $s \neq 0$ and $v(s) = v(y)$. By the first step there is $z \in F$ with $v(s - z) > v(s)$. Thus $v(y - z) > v(y)$ for every $y \in (FL)^\times$, i.e. $(FL/F, v)$ is immediate. □

2.2. **Depth surgery.**

Lemma 2.17. *Assume (H) and let $\mathcal{B}$ be a strong basis for $(F/K, v)$. For every $b \in \mathcal{B} \setminus \{1\}$ there is $n \in \mathbb{N}$ such that $b \notin F^{\times p^n}$.*

Proof. Seeking a contradiction, we suppose that $b \in \mathcal{B} \setminus \{1\}$ satisfies $b \in F^{\times p^n}$ for all n. Applying (SB-IV) to the singleton $B = \{b\}$ with $a_b = 1$, we have $b \in \mathcal{B}^p$: let $b_1 \in \mathcal{B}$ be such that $b = b_1^p$. Note that $b_1 \neq 1$, since otherwise $b = 1$, a contradiction. To iterate, we need to check that $b_1 \in F^{\times p^n}$ for all n. Since $b \in F^{\times p^{n+1}}$, by assumption, there is $c \in F$ with $b = c^{p^{n+1}}$. Thus there exists $\zeta \in F$ with $\zeta^p = 1$ and $b_1 = \zeta c^{p^n}$. In equal characteristic this forces $\zeta = 1 \in K$, whereas in mixed characteristic this forces $\zeta \in K$ by (H). In either case ζ is a p^n-th power in K, so in F, and therefore $b_1 \in F^{\times p^n}$. An iteration shows that $b \in \mathcal{B}^{p^n}$ for all $n \in \mathbb{N}$.

In case (VT), we observe that $v(b) \in \bigcap_n p^n\Gamma_F = \Gamma_K$, since $p^n\Gamma_F = \Gamma_K \oplus p^n\mathbb{Z}\gamma$ (Remark 2.2**(ii)**). Therefore $b = 1$ by (SB-I) and Lemma 2.9**(i)**, which contradicts our choice of b.

In case (RT), we choose $t \in K$ with $v(tb) = 0$. We claim that $\overline{tb} \in k_F^{\times p^n}$, for all $n \in \mathbb{N}$, for which we argue as follows. There is $b_n \in \mathcal{B}$ with $b = b_n^{p^n}$, and so $v(b_n) = p^{-n}v(b) \in \Gamma_K$ (Remark 2.2**(ii)**). Thus there is also $t_n \in K^\times$ with $v(t_n b_n) = 0$. Then for $tb = (tt_n^{-p^n})(t_n b_n)^{p^n}$ we have $\overline{tt_n^{-p^n}} \in k_K^\times = k_K^{\times p^n} \subseteq k_F^{\times p^n}$, since k_K is perfect. Thus $\overline{tb} = \overline{tt_n^{-p^n}}\,\overline{t_n b_n}^{p^n} \in k_F^{\times p^n}$, proving the claim. We also observe that $\bigcap_n k_F^{\times p^n} = k_K^\times$, which holds since k_F/k_K is a regular function field (Remark 2.2**(iii)**) and otherwise an element $u \in \left(\bigcap_n k_F^{\times p^n}\right) \setminus k_K$ gives $[k_K(u^{p^{-n}}) : k_K(u)]$ unbounded inside $k_F/k_K(u)$ which is finite. Putting this together with the claim, we have $\overline{tb} \in k_K$. Finally, let $e \in \mathcal{O}_K^\times$ be such that $\overline{e} = \overline{tb}$, then $v(tb - e) > 0 = \min\{v(tb), v(e)\}$, which contradicts (SB-I) applied to $B = \{1, b\}$ with coefficients $a_b = t$ and $a_1 = -e$. □

Definition 2.18. Assume (H) and let $\mathcal{B}$ be a strong basis of $(F/K, v)$. For $b \in \mathcal{B} \setminus \{1\}$ we write

$$\operatorname{dep}(b) := \max\{n \in \mathbb{N} \mid b \in F^{\times p^n}\}$$

for the **depth** of b, which is well defined by Lemma 2.17.

Lemma 2.19. *Assume (H) and let $\mathcal{B}$ be a strong basis of $(F/K, v)$. Let $b \in \mathcal{B} \setminus \{1\}$ with $\operatorname{dep}(b) \geq 1$. Then there is a unique $e \in \mathcal{B}$ with $b = e^p$, and $\operatorname{dep}(e) = \operatorname{dep}(b) - 1$.*

Proof. Since $\operatorname{dep}(b) \geq 1$ we have $b \in F^{\times p}$, so $b \in \mathcal{B}^p$ by (SB-IV) applied to the singleton $\{b\}$ with coefficient 1. Say $b = e^p$ with $e \in \mathcal{B}$. If also $b = e'^p$ with $e' \in \mathcal{B}$, then $(e/e')^p = 1$, so $e = \zeta e'$ for some p-th root of unity $\zeta \in K^\times$ by (H). As $e, e' \in \mathcal{B}$, this contradicts (SB-I) applied to $\{e, e'\}$ unless $e = e'$.

For the depth, it suffices to show that $e \in F^{\times p^n}$ if and only if $b \in F^{\times p^{n+1}}$, for every $n \in \mathbb{N}$. If $e = g^{p^n}$ then $b = e^p = g^{p^{n+1}}$. Conversely, if $b = g^{p^{n+1}}$ then $e^p = (g^{p^n})^p$, so $e = \zeta g^{p^n}$ for some p-th root of unity $\zeta \in K^\times$. Since K is closed under p-th roots we may write $\zeta = \eta^{p^n}$ with $\eta \in K^\times$, whence $e = (\eta g)^{p^n} \in F^{\times p^n}$. □

The following two lemmas are important for the mixed characteristic case of Proposition 2.22. See the statement of that proposition for the definitions of the properties (MIX-I), (MIX-II), and (MIX-III).

Lemma 2.20 (Removing p-th powers of large value)**.** *Continue to assume (H), suppose that v has mixed characteristic, and let $\mathcal{B}$ be a strong basis of $(F/K, v)$. Let $\mu = 1 + \sum_{b\in B} a_b\, b$ with $B \subseteq \mathcal{B}\setminus\{1\}$ finite and $a_b \in K^\times$, suppose that $\mu \notin F^{\times p}$, and that $v(p) < v(a_b\, b) \leq \frac{p}{p-1}v(p)$ for all $b \in B$. Then there is $\mu_2 = 1 + \sum_{b\in B_2} a_b\, b \in \mu F^{\times p}$ with $B_2 \subseteq \mathcal{B}\setminus\{1\}$ finite and $a_b \in K^\times$, satisfying (MIX-I), (MIX-II) and (MIX-III).*

Proof. Let $b \in B$, and suppose that $b \in \mathcal{B}^p$. By Lemma 2.19, there is a unique $e \in \mathcal{B}\setminus\{1\}$ with $e^p = b$; whence $\operatorname{dep}(e) = \operatorname{dep}(b) - 1$, and there is $d \in K^\times$ such that $a_b = d^p$, by (H). The remaining sum $a := \sum_{b'\in B\setminus\{b\}} a_{b'} b'$ has value $> v(p) \geq \frac{1}{p-1}v(p)$ by (SB-I), and $v((de)^p) = v(a_b\, b) > v(p)$, so Lemma 1.4**(v)** gives $1 + a - p\, de \in (1 + a + a_b\, b)F^{\times p} = \mu F^{\times p}$. Consequently, we may replace the term $a_b\, b$ by $-pde$, whose coefficient $-pd$ lies in $K^\times$ and whose value $v(p) + \frac{1}{p}v(a_b\, b)$ lies in $(v(p), \frac{p}{p-1}v(p)]$. There is a small subtlety: if e already occurs in the sum, we sum the two coefficients. If the resulting term has value $> \frac{p}{p-1}v(p)$, then it may be discarded, since this changes the 1-unit by a factor in $F^{\times p}$, by Lemma 1.4**(iv)**. Each such step replaces an element of depth $\operatorname{dep}(b) \geq 1$ in the support by one of depth $\operatorname{dep}(b) - 1$, and otherwise only deletes elements. Since the multiset ordering on finite multisets of natural numbers is well founded, the process terminates, with $\mu_2 = 1 + \sum_{b\in B_2} a_b\, b \in \mu F^{\times p}$, $B_2 \cap \mathcal{B}^p = \emptyset$, and $v(p) < v(a_b\, b) \leq \frac{p}{p-1}v(p)$ for $b \in B_2$. So (MIX-I) and (MIX-II) hold. As $\mu \notin F^{\times p}$ we have $\mu_2 \neq 1$, so $B_2 \neq \emptyset$, and (MIX-III) holds by (SB-IV), since no dominant element of $\sum_{b\in B_2} a_b\, b$ lies in $\mathcal{B}^p$. □

Lemma 2.21 (Normal form for 1-units)**.** *Continue to assume (H), suppose that v has mixed characteristic, and let $\mathcal{B}$ be a strong basis of $(F/K, v)$. Let μ be a 1-unit of F with $\mu \notin F^{\times p}$. Then there is $\mu_2 = 1 + \sum_{b\in B_2} a_b\, b \in \mu F^{\times p}$ with $B_2 \subseteq \mathcal{B}\setminus\{1\}$ finite and $a_b \in K^\times$, satisfying (MIX-I), (MIX-II) and (MIX-III).*

Proof. Throughout we use Lemma 1.4**(iv)** in the form: two 1-units whose difference has value $> \frac{p}{p-1}v(p)$ lie in the same coset of $F^{\times p}$.

Step 1: renormalisation. We claim: for every 1-unit $1 + w \notin F^{\times p}$ there is $\mu' = 1 + \sum_{b\in B'} a'_b\, b \in (1 + w)F^{\times p}$ satisfying (MIX-I) and (MIX-II), with $v(a'_b\, b) \geq v(w)$ for all $b \in B'$, and μ' satisfies (MIX-III) whenever $\operatorname{rv}(w) = \operatorname{rv}(s)$ for some $s \in K\langle\mathcal{B}\setminus\{1\}\rangle$ having a dominant element outside $\mathcal{B}^p$.

First, $v(w) \leq \frac{p}{p-1}v(p)$, since otherwise $1 + w \in F^{\times p}$. By (SB-II) choose $\nu \in K\langle\mathcal{B}\rangle$ with $v\big((1 + w) - \nu\big) > \frac{p}{p-1}v(p)$. Then ν is a 1-unit and $\nu \in (1 + w)F^{\times p}$. Write $\nu - 1 = (c_1 - 1) + \sum_{b\in D} c_b\, b$ with $D \subseteq \mathcal{B}\setminus\{1\}$ finite, $c_b \in K^\times$, and $c_1 \in K$ the coefficient of 1 in ν. Since $\nu - 1 = \big(\nu - (1 + w)\big) + w$, we have $v(\nu - 1) = v(w) > 0$ and $\operatorname{rv}(\nu - 1) = \operatorname{rv}(w)$. So by (SB-I), $v(c_1 - 1) \geq v(w) > 0$ and $v(c_b\, b) \geq v(w)$ for all $b \in D$. In particular $c_1 \in K^\times = K^{\times p}$, so we may replace ν by $\nu\, c_1^{-1} = 1 + \sum_{b\in D} c_1^{-1} c_b\, b$, and then discard the terms of value $> \frac{p}{p-1}v(p)$. The result μ' satisfies (MIX-I) and (MIX-II), and $v(a'_b\, b) \geq v(w)$ for all its terms, as $v(c_1) = 0$. Suppose moreover that $\operatorname{rv}(w) = \operatorname{rv}(s)$ with s as stated. Then $\operatorname{rv}(\nu - 1) = \operatorname{rv}(s)$, i.e. $v\big((\nu - 1) - s\big) > v(s)$, so by (SB-I) the elements $\nu - 1$ and s have the same dominant elements. Thus 1 is not a dominant element of $\nu - 1$, i.e. $v(c_1 - 1) > v(w)$, and some dominant element $b_0 \in D$ of $\nu - 1$ lies outside $\mathcal{B}^p$. The term of $\mu' - 1$ at b_0 is $c_1^{-1} c_{b_0} b_0$, of value $v(w) \leq \frac{p}{p-1}v(p)$, so it was not discarded, and it is dominant since all terms of $\mu' - 1$ have value $\geq v(w)$. Hence $\operatorname{rv}(\mu' - 1) \notin \mathrm{RV}_F^p$ by (SB-IV), i.e. (MIX-III) holds.

Step 2: removing dominant p-th powers. By Step 1 applied to μ we may assume that $\mu = 1 + \sum_{b\in B_1} a_b\, b$ satisfies (MIX-I) and (MIX-II), and $B_1 \neq \emptyset$ since $\mu \notin F^{\times p}$. If some dominant element of $\mu - 1$ lies outside $\mathcal{B}^p$, then (MIX-III) holds by (SB-IV), and we are done. Otherwise let γ be the least value $v(a_b\, b)$, $b \in B_1$, such that not all $b \in B_1$ with $v(a_b\, b) = \gamma$ lie in $\mathcal{B}^p$, and $\gamma := \infty$ if there is no such value. Write $\mu - 1 = P + Q$, where P is the sum of the terms of value $< \gamma$, and Q the sum of the others. So $P \neq 0$, every basis element in P lies in $\mathcal{B}^p$, and either $Q = 0$, or $v(Q) = \gamma$ and some dominant element of Q lies outside $\mathcal{B}^p$.

For each term $a_b\, b$ of P write $b = e_b^p$ with $e_b \in \mathcal{B}$ and $a_b = d_b^p$ with $d_b \in K^\times$, and put $c_b := d_b\, e_b$, so that $c_b^p = a_b\, b$. The e_b are pairwise distinct, so by (SB-I) the element $t := \sum_b c_b \in K\langle\mathcal{B}\rangle$ has $v(t) = \frac{1}{p}v(P) > 0$. Put $M := (1 + t)^p$ and $\rho := M - 1 - P$, so that

$$\rho = \Big((1 + t)^p - 1 - t^p\Big) + \Big(t^p - \sum_b c_b^p\Big).$$

Each term of the first bracket is $\binom{p}{i}t^i$ with $0 < i < p$, and each term of the second bracket is a multinomial coefficient $\binom{p}{k_1,\dots,k_m}$ with all $k_i < p$ times a product of p of the c_b. In both cases the coefficient is divisible by p and the rest has value $\geq v(t)$. Hence $v(\rho) \geq v(p) + v(t) > v(p)$. Now $\mu = M + Q - \rho$, so $\mu M^{-1} = 1 + w$ with $w := (Q - \rho)M^{-1}$. Here $1 + w \in \mu F^{\times p}$, so $1 + w \notin F^{\times p}$, and $\mathrm{rv}(w) = \mathrm{rv}(Q - \rho)$ since M is a 1-unit.

If $Q \neq 0$ and $v(\rho) > \gamma$, then $\mathrm{rv}(w) = \mathrm{rv}(Q)$, and Step 1 applied to $1 + w$, with $s := Q$, gives the required μ_2. Otherwise $v(w) \geq v(\rho) > v(p)$, so Step 1 applied to $1 + w$ gives μ' satisfying (MIX-I) and (MIX-II) with all terms of value $> v(p)$, and Lemma 2.20 applied to μ' gives the required μ_2. □

2.3. **Simplified minimal polynomials.**

Proposition 2.22 (Simplified minimal polynomials)**.** *Continue to assume* (H)*, and suppose that* $(F/K, v)$ *admits a strong basis* $\mathcal{B}$*. Let* $(E/F, v)$ *be a Galois extension of degree* p*.*

(EQ) *If* v *has equal characteristic, then* E/F *has a minimal polynomial*

$$g = Z^p - Z - a, \qquad a = \sum_{b \in B} a_b b,$$

with $B \subseteq \mathcal{B}$ *finite and* $a_b \in K^\times$*, which is* **simplified***:*

(EQ-I) $B \subseteq (\mathcal{B} \setminus \mathcal{B}^p) \cup \{1\}$,

(EQ-II) $\forall b \in B : v(a_b b) \leq 0$,

(EQ-III) *if* $v(a_b b) < 0$ *for some* $b \in B \setminus \{1\}$*, then* $v(a_1) > \min_{b \in B \setminus \{1\}} v(a_b b)$*, and*

(EQ-IV) *if* $1 \in B$ *and* $v(a_1) < 0$*, then* $v(a_1 - \wp(y)) < 0$ *for all* $y \in K$*.*

(MIX) *If* v *has mixed characteristic, then* E/F *has a minimal polynomial*

$$g = Z^p - xa, \qquad a = 1 + \sum_{b \in B} a_b b,$$

with $x \in K\langle\mathcal{B}\rangle$ *such that* $x = 1$ *or* $\mathrm{rv}(x) \notin \mathrm{RV}_F^p$*,* $B \subseteq \mathcal{B}$ *finite, and* $a_b \in K^\times$*, which is* **simplified***:*

(MIX-I) $B \subseteq \mathcal{B} \setminus \{1\}$,

(MIX-II) $\forall b \in B : 0 < v(a_b b) \leq \frac{p}{p-1} v(p)$*, and*

(MIX-III) $B = \emptyset$ *or* $\mathrm{rv}\big(\sum_{b \in B} a_b b\big) \notin \mathrm{RV}_F^p$*.*

Moreover, in case (VT) *we may take* $x \in (\mathcal{B} \setminus \mathcal{B}^p) \cup \{1\}$*, and in case* (RT) *we may take* $v(x) = 0$*.*

Proof. We begin in case (EQ). Firstly, E/F is an Artin–Schreier extension, so it admits a minimal polynomial of the form $Z^p - Z - \alpha$ for some $\alpha \in F \setminus \wp(F)$, where $\wp(Z) = Z^p - Z$ is the Artin–Schreier polynomial. The extension depends only on the coset $\alpha + \wp(F)$, so we are at liberty to replace α by any other element of $\alpha + \wp(F)$. By (SB-II), there exists $\alpha_0 \in K\langle\mathcal{B}\rangle$ such that $\alpha - \alpha_0 \in \mathfrak{m}_F$. Write $\alpha_0 = \sum_{b \in D_0} a_b b$ for a finite subset $D_0 \subseteq \mathcal{B}$ and coefficients $a_b \in K^\times$. Writing $\alpha_1 := \sum_{b \in D_1} a_b b$, where $D_1 = \{b \in D_0 \mid v(a_b b) \leq 0\}$, we have that $\alpha - \alpha_1 \in \mathfrak{m}_F$. By Lemma 1.4**(i)**, we have $\mathfrak{m}_F \subseteq \wp(F)$, and so $\alpha - \alpha_1 \in \wp(F)$. Given any $b \in D_1 \cap (\mathcal{B}^p \setminus \{1\})$, we have $a_b b \in F^{\times p}$ (since $a_b \in K^\times = K^{\times p}$ by (H)). Letting $\alpha_2 := \alpha_1 - a_b b + (a_b b)^{1/p}$, we have $\alpha_1 - \alpha_2 \in \wp(F)$. We can also view this as a construction of the set $D_2 := (D_1 \setminus \{b\}) \cup \{b^{1/p}\}$ – if $b^{1/p}$ is in D_1, then the coefficients will add, but this causes no problem. By Lemma 2.17, for each $b \in \mathcal{B} \setminus \{1\}$ there is $n \in \mathbb{N}$ such that $b \notin F^{\times p^n}$. Iterating this process finitely many times, we construct a finite set $B \subseteq (\mathcal{B} \setminus \mathcal{B}^p) \cup \{1\}$, and coefficients $a_b \in K^\times$, such that the element $a = \sum_{b \in B} a_b b$ satisfies both (EQ-I) and (EQ-II), and also $\alpha - a \in \wp(F)$. Thus we may assume the minimal polynomial is simplified. It remains to arrange (EQ-III). If $1 \notin B$, or if $v(a_b b) = 0$ for all $b \in B \setminus \{1\}$, there is nothing to prove (reading $v(a_1) = \infty$ if $1 \notin B$). So suppose that $1 \in B$ and that $\gamma := \min_{b \in B \setminus \{1\}} v(a_b b) < 0$. If $v(a_1) = 0$ then $v(a_1) > \gamma$ already, so suppose that $v(a_1) < 0$, using (EQ-II). Since K is closed under p-th roots by (H), we have $a_1^{1/p} \in K^\times$ and $a_1 - a_1^{1/p} = \wp(a_1^{1/p}) \in \wp(K)$. Iterating, $a_1 - a_1^{1/p^\nu} \in \wp(K)$ and $v(a_1^{1/p^\nu}) = p^{-\nu} v(a_1) < 0$ for every $\nu \in \mathbb{N}$. Since v has rank 1 by (H), the value group is archimedean, so there is ν with $p^{-\nu} v(a_1) > \gamma$. Replacing a_1 by a_1^{1/p^ν} changes a by an element of $\wp(K) \subseteq \wp(F)$, leaves B and the other coefficients unchanged, so preserves (EQ-I) and (EQ-II), and now $v(a_1) > \gamma$, which is (EQ-III). (This is the replacement used by Kuhlmann in the proof of [Kuh10a, Proposition 4.12].) Finally we arrange (EQ-IV). Suppose that $1 \in B$, that $v(a_1) < 0$, and that there is $y \in K$ with $v(a_1 - \wp(y)) \geq 0$ (otherwise (EQ-IV) already holds). Put $a_1' := a_1 - \wp(y)$. Replacing a_1 by a_1' changes a by an element of $\wp(K) \subseteq \wp(F)$ and leaves $B \setminus \{1\}$ and its coefficients unchanged. If $v(a_1') = 0$, we keep $1 \in B$ with the new coefficient $a_1' \in K^\times$. If $v(a_1') > 0$ (including the case $a_1' = 0$), then $a_1' \in \mathfrak{m}_K \subseteq \wp(K)$ by Lemma 1.4**(i)**, so we may remove 1 from B. The resulting B is nonempty, since otherwise $a \in \wp(F)$, contradicting the irreducibility of the minimal polynomial. In both cases (EQ-I) and (EQ-II) still hold. Condition (EQ-III) still holds, since $v(a_1)$ has only increased, from a negative value to 0 or to ∞. Finally, (EQ-IV) now holds, vacuously.

We turn to case (MIX). Firstly, E/F is a Kummer extension of degree p, and K (thus also F) contains all p-th roots of unity by (H). So E/F admits a minimal polynomial of the form $Z^p - \alpha$ for some $\alpha \in F^\times \setminus F^{\times p}$. The extension depends only on the coset $\alpha F^{\times p}$, so we are at liberty to replace α by any other element of $\alpha F^{\times p}$.

Claim 2.22.1. The coset $\alpha F^{\times p}$ contains a product $x\mu$, where μ is a 1-unit and $x \in K\langle\mathcal{B}\rangle$ satisfies either $x = 1$ or $\mathrm{rv}(x) \notin \mathrm{RV}_F^p$. Moreover, in case (VT) we may take $x \in (\mathcal{B} \setminus \mathcal{B}^p) \cup \{1\}$, and in case (RT) we may take $v(x) = 0$.

Proof of claim. We treat the two cases (VT) and (RT) separately. First, suppose we are in case (VT). By Lemma 2.8**(ii)** there is $\beta_0 \in \mathcal{B}$ with $v(\alpha) - v(\beta_0) \in \Gamma_K$. We choose $\gamma \in K^\times$ with $v(\gamma) = v(\alpha) - v(\beta_0)$. Since $k_F = k_K$, there is $\delta \in \mathcal{O}_K^\times$ with $\overline{\delta} = \overline{\alpha\beta_0^{-1}\gamma^{-1}}$. Then $\mu := \alpha\beta_0^{-1}\gamma^{-1}\delta^{-1}$ is a 1-unit. Since K is closed under p-th roots, both γ and δ lie in $K^\times = K^{\times p} \subseteq F^{\times p}$, and so $\beta_0\mu \in \alpha F^{\times p}$. If $\beta_0 \in \mathcal{B}^p$, then $\beta_0 \in F^{\times p}$ and we take $x := 1$. Otherwise we take $x := \beta_0$, and then $\mathrm{rv}(x) \notin \mathrm{RV}_F^p$ by (SB-IV) applied to the singleton $\{x\}$ with $a_x = 1$. In either case $x\mu \in \alpha F^{\times p}$.

Suppose next that we are in case (RT). Since $\Gamma_F = \Gamma_K$, there is $\gamma \in K^\times$ with $v(\gamma) = v(\alpha)$. Moreover $\gamma \in K^\times = K^{\times p} \subseteq F^{\times p}$, since K is closed under p-th roots. Write $u := \overline{\alpha\gamma^{-1}} \in k_F^\times$. If $u \in k_F^{\times p}$, say $u = w^p$, then by Lemma 2.8**(iii)** there is $y \in K\langle\mathcal{B}\rangle$ with $v(y) = 0$ and $\overline{y} = w$. Thus $\mu := \alpha\gamma^{-1}y^{-p}$ is a 1-unit, and we take $x := 1$. Otherwise, by Lemma 2.8**(iii)** there is $x \in K\langle\mathcal{B}\rangle$ with $v(x) = 0$ and $\overline{x} = u$. Thus $\mu := \alpha\gamma^{-1}x^{-1}$ is a 1-unit, and $\mathrm{rv}(x) \notin \mathrm{RV}_F^p$ by Lemma 2.5**(ii)**, since $\overline{x} = u \notin k_F^{\times p}$. In either case $x\mu \in \alpha F^{\times p}$. ■claim

If $\mu \in F^{\times p}$ then $\alpha F^{\times p} = xF^{\times p}$, so $g = Z^p - x$ is a minimal polynomial of the required form, with $B = \emptyset$, and (MIX-I), (MIX-II) hold vacuously while (MIX-III) holds trivially (by its first disjunct). We may therefore assume that $\mu \notin F^{\times p}$.

Claim 2.22.2. The coset $\alpha F^{\times p}$ contains a product $x\mu_2$ where $\mu_2 = 1 + \sum_{b\in B_2} a_b b$ for a finite set $B_2 \subseteq \mathcal{B} \setminus \{1\}$ and coefficients $a_b \in K^\times$. Moreover (MIX-I), (MIX-II), and (MIX-III) hold for the sum $\mu_2 = 1 + \sum_{b\in B_2} a_b b$.

Proof of claim. By Lemma 2.21 applied to μ, we obtain $\mu_2 = 1 + \sum_{b\in B_2} a_b b \in \mu F^{\times p}$ satisfying (MIX-I), (MIX-II) and (MIX-III). Since $\alpha \in x\mu F^{\times p}$ by Claim 2.22.1, we conclude $\alpha \in x\mu_2 F^{\times p}$. ■claim

By Claim 2.22.2 we may take $a := \mu_2$. □

Remark 2.23 (Historical remarks)**.** The normal forms of this section are old. In equal characteristic, the "reduced form" of an Artin–Schreier generator (i.e. poles of order prime to p, obtained by replacing a leading term c^p by c, which requires p-th roots in the constant field) is due to Hasse [Has34, §2, (5)–(7)], who also reads the ramification of a prime off from the reduced form [Has34, §3 b)]. The same reduction appears in [Kuh10b, Remark 2.29] and [FV02, III (2.4)]. Condition (EQ-III) and the twist $\beta = d^{-1}\alpha$ of case (RT-INSEP) have no analogue over a perfect residue field, cf. [Kuh04, Lemma 2.13]. In mixed characteristic, Lemma 1.4**(iv),(v)** and the procedure of Lemmas 2.20 and 2.21 follow [Kuh10a, §2.2 and Propositions 4.6, 4.13]. Conditions (MIX-I)–(MIX-III) are the strong-basis analogue of the classical description of the 1-units of a local field modulo p-th powers by the elements $1 + \theta\pi^i$, $1 \le i < \frac{p}{p-1}e$, $p \nmid i$, together with one element of level $\frac{p}{p-1}e$ [FV02, I (5.8), (6.4)–(6.5)]. The substitution $Z = \xi Y + 1$ with $\xi^{p-1} = -p$, which turns a Kummer extension of degree p into one of Artin–Schreier type, goes back to MacKenzie and Whaples [MW56] (see [FV02, III §2]). Its group-scheme incarnation, with $\lambda = \zeta_p - 1$ in place of ξ, is the deformation of $\mathbb{G}_a$ to $\mathbb{G}_m$ of [Wat87, SOS89].

2.4. **Ramification of Galois degree-p extensions.**

Proposition 2.24 (Ramification of Galois degree-p extensions)**.** *Continue to assume (H), and suppose that $(F/K, v)$ admits a strong basis $\mathcal{B}$. Let $(E/F, v)$ be a Galois extension of degree p with simplified minimal polynomial g as in Proposition 2.22: in case (EQ), $g = Z^p - Z - a$ with $a = \sum_{b\in B} a_b b$, and in case (MIX), $g = Z^p - xa$ with $a = 1 + \sum_{b\in B} a_b b$. Exactly one of the following five cases holds.*

(SPLIT) *Suppose that (EQ) holds and $B = \{1\}$, i.e. the simplified minimal polynomial g is defined over K. Then E is the compositum of F and an Artin–Schreier extension E_0 of K, and $d(E/F) = d(E_0/K)$.*

(VT-RAM) *Suppose that (VT) holds and, if (EQ) holds, $B \neq \{1\}$. Then E/F is totally ramified: $(\Gamma_E : \Gamma_F) = p$.*

The remaining three cases occur when (RT) holds and, if (EQ) holds, $B \neq \{1\}$:

(RT-INSEP) *Suppose that*
- (EQ) *$v(a_b b) < 0$ for some $b \in B \setminus \{1\}$;*
- (MIX) *$x \neq 1$, or $v(a_b b) \neq \frac{p}{p-1}v(p)$ for some $b \in B$.*

Then E/F is unramified, and k_E/k_F is purely inseparable of degree p.

(RT-SEP) *Suppose that*
- (EQ) *$v(a_b b) = 0$ for all $b \in B$;*
- (MIX) *$x = 1$, and $v(a_b b) = \frac{p}{p-1}v(p)$ for all $b \in B$.*

Then E/F is inert, and k_E/k_F is separable of degree p.

(RT-BASE) *Suppose that* (EQ) *holds, $1 \in B$ with $v(a_1) < 0$, and $v(a_b b) = 0$ for all $b \in B \setminus \{1\}$. Let α' be a root of $Z^p - Z - a_1$, and put $K' := K(\alpha')$. Then E/F and K'/K are both immediate, so that $d(E/F) = d(K'/K) = p$. Moreover, EK'/FK' is defectless.*

In case (SPLIT)*, if (K, v) is defectless then so is $(E/F, v)$. In cases* (VT-RAM)*,* (RT-INSEP) *and* (RT-SEP)*, $(E/F, v)$ is defectless. In case* (RT-BASE)*, (K, v) is not defectless.*

Proof. Each of the five cases is a hypothesis (usually "Suppose…", though the latter three also hypothesize that (RT) holds, and if (EQ) then $B \neq \{1\}$) followed by a conclusion ("Then…"). We first argue that the five cases are pairwise exclusive, by showing that their hypotheses contradict one another: (VT) versus (RT), $B = \{1\}$ versus $B \neq \{1\}$, and the conditions on the values $v(a_b b)$. Next we argue that at least one case's hypothesis holds. By Remark 2.2**(iii)**, and since we have assumed (H), either (VT) or (RT) holds. If (EQ) holds and $B = \{1\}$, we have satisfied the hypothesis of (SPLIT). Otherwise, in case (VT) we are right away in case (VT-RAM). In case (RT) with (MIX), the hypotheses of (RT-INSEP) and (RT-SEP) are complementary. In case (RT) with (EQ) and $B \neq \{1\}$: if $v(a_b b) < 0$ for some $b \in B \setminus \{1\}$ then we satisfy the hypotheses of (RT-INSEP). Otherwise $v(a_b b) = 0$ for all $b \in B \setminus \{1\}$ by (EQ-II), and we satisfy the hypotheses of (RT-BASE) if $1 \in B$ and $v(a_1) < 0$, and of (RT-SEP) if not, using $v(a_1) \leq 0$ from (EQ-II).

It remains to check each case as a straightforward implication "hypothesis $\implies$ conclusion". In each case we will follow the same pattern, composed of three steps. For the first step, no matter whether in mixed or equal characteristic, we twist the extension into Artin–Schreier form. For the second step, we make several short observations on twisted Artin–Schreier form. The third step is almost trivial by this point: we deduce from the Fundamental Equality that if $e(E/F, v)$ or $f(E/F, v)$ is p, then $d(E/F, v) = 1$.

Throughout, α is a root of g, so that $E = F(\alpha)$, and $B_0 \subseteq B$ is the set of dominant basis elements of $\sum_{b \in B} a_b b$.

(A) **The Artin–Schreier twist.** In case (EQ), put $\beta := \alpha$, $c_b := a_b$ for $b \in B$, and $e_i := 0$ for $2 \leq i \leq p-1$. On the other hand, in case (MIX) with $x = 1$, we denote $\beta := \xi^{-1}(\alpha - 1)$, with ξ as in Lemma 1.4**(iii)**, $c_b := \xi^{-p} a_b$ for $b \in B$, and $e_i := \binom{p}{i} \xi^{i-p}$ for $2 \leq i \leq p-1$. In both cases let

$$c := \sum_{b \in B} c_b b, \qquad\qquad h(Y) := Y^p - Y + \sum_{i=2}^{p-1} e_i Y^i - c \in F[Y].$$

Thus in case (EQ), we have merely renamed the data $(\beta, c, h(Y)) := (\alpha, a, g(Y))$, whereas in case (MIX), we have twisted the data $(\beta, c, h(Y)) := (\xi^{-1}(\alpha - 1), \xi^{-p} a, \xi^{-p} g(\xi Y + 1))$. We make four little observations:

- **(i)** $h(\beta) = 0$, $E = F(\beta)$, and h is the minimal polynomial of β over F. In case (MIX) this follows by substituting $\alpha = \xi\beta + 1$ in $\alpha^p = a$ and dividing by ξ^p, using $p\,\xi^{1-p} = -1$.
- **(ii)** $v(e_i) = (i-1)v(\xi) > 0$ for $2 \leq i \leq p-1$ in case (MIX), so all e_i are elements of $\mathfrak{m}_K$.
- **(iii)** $B \neq \emptyset$, since otherwise g would have the root 0 (case (EQ)) or 1 (case (MIX)). The dominant elements of $\sum_b c_b b$ are those of B_0, and $v(c) = \min_{b \in B} v(c_b b) \leq 0$, by (SB-I) together with (EQ-II), respectively with (MIX-II) since $v(\xi^{-p}) = -\frac{p}{p-1} v(p)$.
- **(iv)** $\mathrm{rv}(c) \notin \mathrm{RV}_F^p$, provided $1 \notin B_0$ in case (EQ). Indeed, in case (EQ) we have $B_0 \subseteq B \setminus \{1\} \subseteq \mathcal{B} \setminus \mathcal{B}^p$ by (EQ-I), so Remark 2.12 applies to $c = a$. In case (MIX), $\mathrm{rv}(c) = \mathrm{rv}(\xi^{-1})^p \mathrm{rv}\big(\sum_b a_b b\big)$ and $\mathrm{rv}\big(\sum_b a_b b\big) \notin \mathrm{RV}_F^p$ by (MIX-III).

In terms of c rather than a, the conditions of the proposition read the same in both characteristics: since $v(c_b b) = v(a_b b) - \frac{p}{p-1} v(p)$ in case (MIX), the condition $v(a_b b) < \frac{p}{p-1} v(p)$ (respectively $v(a_b b) = \frac{p}{p-1} v(p)$), becomes $v(c_b b) < 0$ (respectively $v(c_b b) = 0$), as in case (EQ). Moreover $1 \notin B$ in case (MIX), by (MIX-I), which is why (SPLIT) and (RT-BASE) occur only in case (EQ).

(B) **Simple valuation calculations with the twisted Artin–Schreier polynomial.** With the data (β, c, h) as in (A):

- **(a)** If $v(\beta) < 0$, then $v(c) = pv(\beta) < 0$. If $v(\beta) > 0$, then $v(c) = v(\beta) > 0$. If $v(\beta) = 0$, then $v(c) \geq 0$.
- **(b)** If $v(c) < 0$ and $d \in K^\times$ satisfies $pv(d) = -v(c)$, then $\chi := d\beta$ satisfies $v(\chi) = 0$ and $\overline{\chi}^p = \overline{d^p c}$.
- **(c)** If $v(c) = 0$, then $v(\beta) = 0$, and $\overline{\beta}$ generates a separable extension of k_F of degree p.

For **(a)**: these are simple arguments using the Ultrametric Inequality. If $v(\beta) < 0$ then $v(\beta^p) < v(\beta) < v(e_i \beta^i)$, for every $i \in \{2, \ldots, p-1\}$. Thus β^p is the unique term of minimum value in the sum $\beta^p - \beta + \sum_i e_i \beta^i$. Thus $v(c) = v(\beta^p) = pv(\beta)$, which is negative. On the other hand if $v(\beta) > 0$ then $v(\beta) < v(\beta^p)$ and $v(\beta) < v(e_i \beta^i)$, for every $i \in \{2, \ldots, p-1\}$. Thus β is the unique term of minimum value in the sum $\beta^p - \beta + \sum_i e_i \beta^i$. Thus

$v(c) = v(\beta)$, which is positive. In the third case, if $v(\beta) = 0$ then $v(\beta^p) = 0$ and $v(e_i\beta^i) > 0$, for every $i \in \{2, \dots, p-1\}$. For **(b)**: by **(a)**, $v(d) = -v(\beta) > 0$, so $v(\chi) = 0$. Multiplying the equation $h(\beta) = 0$ by d^p yields

$$\chi^p - d^{p-1}\chi + \sum_{i=2}^{p-1} e_i d^{p-i}\chi^i = d^p c.$$

Since $v(d^{p-1}) > 0$, $v(e_i d^{p-i}) > 0$, and $v(d^p c) = 0$, by taking residues we find $\overline{\chi}^p = \overline{d^p c}$. For **(c)**: by **(a)**, $v(c) = 0$ implies $v(\beta) = 0$. Thus we have $\overline{h} = Y^p - Y - \overline{c} \in k_F[Y]$, by (A)**(ii)**, of which $\overline{\beta}$ is a root. This polynomial is separable, and it has no root in k_F: a root would be simple, so by henselianity h would have a root in F, contradicting (A)**(i)**. An Artin–Schreier polynomial without a root is irreducible, so $k_F(\overline{\beta})/k_F$ is separable of degree p.

Throughout the rest of the proof we write $e = e(E/F)$, $f = f(E/F)$, and $d = d(E/F)$. Since F is henselian and $[E : F] = p$, the Fundamental Equality gives $p = efd$: each of the three is 1 or p, and exactly one of them is p. So if $e > 1$ or $f > 1$, then that invariant is p, the other is 1, and $d = 1$.

It remains to prove each of the five cases separately, passing several times via **(A)** and **(B)**.

Case (SPLIT). Here we have (EQ) and $B = \{1\}$, so in particular (A) applies. Since $a = a_1 \in K$, the extension E/F splits over K: since $a_1 \notin \wp(F) \supseteq \wp(K)$, the extension $E_0 := K(\alpha)/K$ has degree p, and $E = F(\alpha) = FE_0$, i.e. E is the compositum of F/K and E_0/K. We write $d_0 = d(E_0/K)$, $e_0 = e(E_0/K)$, and $f_0 = f(E_0/K)$. To finish this case it simply remains to show that $d_0 = d$.

We recall that $v(a_1) \leq 0$, by (EQ-II). If $v(a_1) = 0$ then (B)**(c)**, applied to E/F and to E_0/K (both F and K are henselian), gives $f = f_0 = p$, and so $e = e_0 = 1$. If $v(a_1) < 0$, then by (EQ-IV) and Lemma 1.6, E_0/K is immediate and $d(E_0/K) = p$. By Lemma 2.16 with $L = E_0$, E/F is immediate too, so $e = f = 1$ and $d(E/F) = p = d(E_0/K)$. In either case $e = e_0$ and $f = f_0$, so $d = d_0$. ■case

Case (VT-RAM). Here we have (VT) and, if (EQ) then $B \neq \{1\}$. Thus $k_F = k_K$, which is perfect, by Remark 2.2**(i,iii)**. Suppose first that (A) does not apply, i.e. (MIX) holds and $x \neq 1$. Then $\mathrm{rv}(x) \notin \mathrm{RV}_F^p$ since g is simplified. By Lemma 2.5**(iii)**, $v(x) \notin p\Gamma_F$. As a is a 1-unit, $v(\alpha) = \frac{1}{p}v(x) \notin \Gamma_F$. Therefore $e > 1$, and so $e = p$ and $f = d = 1$. Otherwise, (A) applies, i.e. either (EQ) holds, or (MIX) holds and $x = 1$. In case (EQ), every $b \in B \setminus \{1\}$ has $v(b) \notin \Gamma_K$ by Lemma 2.9**(i)**, so $v(a_b b) \neq 0$, and hence $v(a_b b) < 0$ by (EQ-II). As $B \neq \{1\}$, (EQ-III) applies and gives $1 \notin B_0$. So in both characteristics $\mathrm{rv}(c) \notin \mathrm{RV}_F^p$ by (A)**(iv)**, and $v(c) \notin p\Gamma_F$ by Lemma 2.5**(iii)**. In particular $v(c) \neq 0$, so $v(c) < 0$ by (A)**(iii)**, so $v(\beta) < 0$ by (B)**(a)** (if $v(\beta) \geq 0$ then $v(c) \geq 0$), and then $pv(\beta) = v(c)$, again by (B)**(a)**. Hence $v(\beta) \notin \Gamma_F$. Therefore $e > 1$, and so $e = p$ and $f = d = 1$. In particular E/F is totally ramified and defectless. ■case

In the three following cases, we have (RT), thus $\Gamma_F = \Gamma_K$, which is p-divisible, by Remark 2.2**(ii,iii)**.

Case (RT-INSEP). Suppose first that (MIX) holds and $x \neq 1$, so (A) does not apply. In case (RT), we have $v(x) = 0$ by the final claim of Proposition 2.22. Since also $\mathrm{rv}(x) \notin \mathrm{RV}_F^p$, we have $\overline{x} \notin k_F^{\times p}$ by Lemma 2.5**(ii)**. Since a is a 1-unit, $v(\alpha) = 0$ and $\overline{\alpha}^p = \overline{x}$. This establishes that $\overline{\alpha} \in k_E \setminus k_F$, while $\overline{\alpha}^p \in k_F$. Therefore k_E/k_F is purely inseparable, and $f = p$ and $e = d = 1$. Otherwise (A) applies, and thus that $v(c_b b) < 0$ for some $b \in B$, with $b \neq 1$ in case (EQ). Then $1 \notin B_0$: in case (EQ) by (EQ-III), and in case (MIX) by (MIX-I). So $v(c) < 0$ by (A)**(iii)**, and $\mathrm{rv}(c) \notin \mathrm{RV}_F^p$ by (A)**(iv)**. Choose $d \in K^\times$ with $pv(d) = -v(c)$. By (B)**(b)**, $\chi := d\beta \in E$ has $v(\chi) = 0$ and $\overline{\chi}^p = \overline{d^p c}$. Since $v(d^p c) = 0$ and $\mathrm{rv}(d^p c) = \mathrm{rv}(d)^p \mathrm{rv}(c) \notin \mathrm{RV}_F^p$, Lemma 2.5**(ii)** gives $\overline{d^p c} \notin k_F^{\times p}$, so $\omega := \overline{\chi} \notin k_F$, while $\omega^p \in k_F$. This establishes that k_E/k_F is purely inseparable of degree p, so $f = p$ and $e = d = 1$. In particular E/F is unramified and defectless. ■case

Case (RT-SEP). Here we have $x = 1$ in case (MIX), so (A) applies. By hypothesis $v(c_b b) = 0$ for all $b \in B$, so $v(c) = 0$ by (A)**(iii)**. By (B)**(c)**, $k_E \supseteq k_F(\overline{\beta})$, which is separable of degree p over k_F. Therefore k_E/k_F itself is separable of degree p, so $f = p$ and $e = d = 1$. In particular E/F is inert and defectless. ■case

Case (RT-BASE). We write $d_0 = d(K'/K)$, $e_0 = e(K'/K)$, and $f_0 = f(K'/K)$. Since $\Gamma_K = \Gamma_F$ is p-divisible, by Remark 2.2**(ii,iii)**, $e = 1$ and $e_0 = 1$. By (EQ-IV) and Lemma 1.6, K'/K is immediate of degree p, so $d_0 = p$. Moreover, $\alpha' \notin F$ by (EQ-IV) and Lemma 2.13. Thus $Z^p - Z - a_1$ is irreducible over F, which implies that $[FK' : F] = [K' : K] = p$. Furthermore FK'/F is immediate by Lemma 2.16[1]. So $d(FK'/F) = p$. The element $\alpha'' := \alpha - \alpha'$ is a root of $g'' := Z^p - Z - a''$, where $a'' := \sum_{b \in B \setminus \{1\}} a_b b$, and $EK' = FK'(\alpha'')$. Since $B \neq \{1\}$, (SB-I) and the hypotheses give $v(a'') = 0$, and therefore $v(\alpha'') = 0$, as in (B)**(a)**. If $\overline{a''} \in \wp(k_{FK'})$, then g'' has a root in FK' by henselianity. The roots of g'' differ by elements of $\mathbb{F}_p$, so it follows that $\alpha'' \in FK'$

[1]This argument appears already in case (SPLIT): $Z^p - Z - a_1$ is a simplified minimal polynomial for FK'/F in case (SPLIT).

and $EK' = FK'$. (In fact this case never occurs: $\alpha'' \notin FK'$, so EK'/FK' is an inert extension of degree p, see Remark 2.26**(i)**. We do not need this observation for the present proposition.) Otherwise $Y^p - Y - \overline{a''}$ is irreducible over $k_{FK'}$, since it is an Artin–Schreier polynomial without a root in $k_{FK'}$, and $\overline{\alpha''}$ is a root, so $[k_{EK'} : k_{FK'}] \geq p \geq [EK' : FK']$. Either way EK'/FK' is trivial or inert, and $d(EK'/FK') = 1$. By multiplicativity of defect, we have

$$d(EK'/E) \cdot d = d(EK'/F) = d(EK'/FK') \cdot d(FK'/F) = p.$$

It remains to rule out the possibility that $d(EK'/E) = p$. Suppose, seeking a contradiction, that indeed $d(EK'/E) = p$, so $d = 1$, $[EK' : E] = p$, and $f(EK'/E) = 1$. We want to show that $\alpha' \in E$. Since $v(\alpha'') = 0$, we have $\overline{\alpha''} \in k_{EK'}$, and $k_{EK'} = k_E$ since $f(EK'/E) = 1$. Thus there is some $\varepsilon \in E$ with $\overline{\varepsilon} = \overline{\alpha''}$. Let $\delta := \varepsilon + \alpha' - \alpha = \varepsilon - \alpha''$. Observe that $v(\delta) > 0$ and that $\wp(\delta) = \wp(\varepsilon - \alpha) + a_1 \in E$. Then $v(\wp(\delta)) = v(\delta) > 0$, so $\wp(\delta) \in \mathfrak{m}_E \subseteq \wp(E)$ by Lemma 1.4**(i)**. Therefore $\delta \in E$, since the roots of $Z^p - Z - \wp(\delta)$ differ by elements of $\mathbb{F}_p$. This implies that $\alpha' \in E$ and so $K' \subseteq E$, then $EK' = E$, contradicting $[EK' : E] = p$. Hence $d = p = d_0$. Since $[E : F] = p$, this means that E/F is immediate, as is K'/K. ■case

The final three sentences of the statement are now immediate. □

Remark 2.25. For the proof of Theorem 1, we will apply Proposition 2.24 only when K is algebraically closed. Step 2 reduces to this case, and the proposition enters only through Lemma 2.27, inside Proposition 2.30. In that situation the cases (SPLIT) and (RT-BASE) cannot occur, since $\wp(K) = K$: in case (SPLIT) we would have $a = a_1 \in \wp(K)$, so that g would have a root in F. In case (RT-BASE) we would have $1 \in B$ and $v(a_1) < 0$ with $a_1 \in \wp(K)$, contradicting (EQ-IV). Thus the proof of Theorem 1 uses only the cases (VT-RAM), (RT-INSEP) and (RT-SEP), and does not need Lemmas 1.6, 2.16 and 2.13, which serve only the cases (SPLIT) and (RT-BASE). We have nevertheless stated and proved the proposition in the generality of (H), since it is of independent interest.

Remark 2.26 (Alternative arguments for (RT-BASE))**.** The proof of the case (RT-BASE) above has two steps: **(I)** EK'/FK' is defectless, which we deduced from Hensel's Lemma using only $v(a'') = 0$, and **(II)** E/F is immediate, which we deduced by examining the residue fields and using the multiplicativity of defect. Both of these steps can be done differently.

- **(i) $EK' \neq FK'$, and (I) via "additive rigidity".** By (EQ-I), we have $B \subseteq (\mathcal{B} \setminus \mathcal{B}^p) \cup \{1\}$, so Lemma C.1 gives $a'' \notin \wp(F) + \mathfrak{m}_F$, i.e. $\overline{a''} \notin \wp(k_F)$. Since FK'/F is immediate, $k_{FK'} = k_F$. If $\alpha'' \in FK'$, then taking residues in $\wp(\alpha'') = a''$ would give $\overline{a''} = \wp(\overline{\alpha''}) \in \wp(k_F)$, which is impossible. Hence $\alpha'' \notin FK'$, which shows that the first alternative in the proof of case (RT-BASE) never occurs. More precisely, $Y^p - Y - \overline{a''}$ is irreducible over $k_{FK'}$, so EK'/FK' is inert of degree p, which yields **(I)**. Moreover g'' is irreducible over F, so it is a simplified minimal polynomial of $F(\alpha'')/F$, which falls under case (RT-SEP).
- **(ii) (I) via base change of the strong basis.** Since K'/K is immediate, $k_{K'} = k_K$. If $v(b) = 0$ for all $b \in \mathcal{B}$, as we may arrange by Proposition 2.14**(ii)** (or directly: since $\Gamma_F = \Gamma_K$ in case (RT), rescale each $b \in \mathcal{B}$ by some $t_b \in K^\times$ so that $v(t_b b) = 0$, choosing $t_1 = 1$ and $t_{b^p} = t_b^p$ along each chain $b, b^p, b^{p^2}, \ldots$ that starts at some $b \in \mathcal{B} \setminus \mathcal{B}^p$, by Lemma 2.17. This preserves (SB-I)–(SB-III), and changes the coefficients of a'' only by factors in $K^\times$), then $\mathcal{B}$ is also a strong basis of $(FK'/K', v)$, by Lemma C.5. By **(i)**, g'' is irreducible over FK', so it is a simplified minimal polynomial of EK'/FK' over the base K', which falls under case (RT-SEP). Hence EK'/FK' is inert.
- **(iii) (II) via best approximations.** By (EQ-IV) and Lemma 2.13, $a_1 \notin \wp(F) + \mathcal{O}_F$, so $v(\alpha' - y) < 0$ for all $y \in F$ (otherwise $v(a_1 - \wp(y)) = v(\wp(\alpha' - y)) \geq 0$). Since $v(\alpha'') = 0$, it follows that $v(\alpha - y) = v(\alpha' - y)$ for all $y \in F$. As FK'/F is immediate, the set $\{v(\alpha' - y) \mid y \in F\}$ has no maximum, by Observation 1.1**(i)**, and therefore neither does $\{v(\alpha - y) \mid y \in F\}$. On the other hand, if E/F were not immediate, then $[E : F] = p = e(E/F)f(E/F)$, and α would have a best approximation in F, by Observation 1.1**(ii)**. Hence E/F is immediate. This avoids both step **(I)** and the multiplicativity of defect, although **(I)** is still needed for the claim that EK'/FK' is defectless.

2.5. **Proof of Theorem 1.** The proof of Theorem 1 follows Kuhlmann [Kuh10a, §5]: Steps 1–3 below correspond to [Kuh10a, Lemmas 5.1–5.4], reorganized, with Step 2 combining Lemmas 5.2 and 5.3, Lemma 2.1 is [Kuh10a, Lemma 2.24], Lemma 2.28 is [Kuh10a, Lemma 2.27], and Proposition 2.30 combines [Kuh10a, §5 and Lemma 5.5]. The new ingredient is Lemma 2.27, which replaces [Kuh10a, Corollary 4.2] by Proposition 2.24.

The proof has two halves. First, Steps 1–3 below reduce Theorem 1 to the core case (H_{AC}), in which K is algebraically closed and $(F/K, v)$ is a henselized function field of transcendence degree 1 and rank 1 (Proposition 2.30). Second, the fields satisfying (H_{AC}) are closed under finite extensions, so by **(D3)** it suffices to show

that each of them is algebraically maximal. This follows from Lemma 2.27, which rules out immediate extensions of degree p of strongly inertially generated fields, together with Lemma 2.28, which keeps us among strongly inertially generated fields.

We use the following standard facts.

(D1) (K,v) is [separably] defectless if and only if its henselization (K^{h},v) is [separably] defectless [End72, Theorem (18.2)].

(D2) A finite extension of a defectless field is defectless (by the multiplicativity of the defect, see remarks at the beginning of Section 1).

(D3) Let (K,v) be henselian of residue characteristic $p>0$, and let L/K be finite. Then there is a finite extension N/K of degree prime to p such that $d(L/K)=d(LN/N)$ and LN/N is a tower $N=N_0\subset N_1\subset\cdots\subset N_m=LN$ of normal extensions of degree p. Consequently, if every finite extension of K is algebraically maximal, then (K,v) is defectless. To see this, let L_s be the separable closure of K in L, and let M be the Galois closure of L_s/K. Choose a Sylow p-subgroup Q of $\mathrm{Gal}(M/L_s)$ and a Sylow p-subgroup P of $\mathrm{Gal}(M/K)$ containing Q, and put $N:=M^P$. Then $P\cap\mathrm{Gal}(M/L_s)=Q$, so $[N:K]=(\mathrm{Gal}(M/K):P)$ and $[L_sN:L_s]=(\mathrm{Gal}(M/L_s):Q)$ are prime to p, and so is $[LN:L]=[L_sN:L_s]$, as L/L_s is purely inseparable. For a finite extension L'/K' of henselian fields, $d(L'/K')$ is a power of p and divides $[L':K']=d(L'/K')\,e(L'/K')\,f(L'/K')$. Hence $d(N/K)=d(LN/L)=1$, and multiplicativity gives $d(L/K)=d(LN/K)=d(LN/N)$. As $Q\le P$ are p-groups, there is a chain $Q=P_0<P_1<\cdots<P_k=P$ in which each P_i is normal of index p in P_{i+1}. The fixed fields form a tower of Galois extensions of degree p from N to $L_sN=M^Q$, and LN/L_sN is a tower of purely inseparable extensions of degree p. Finally, if $d(L/K)>1$, then some step N_{i+1}/N_i has defect p, hence $e=f=1$, and so N_i, which is finite over K, is not algebraically maximal.

(D4) Let $v=w\circ\bar{w}$ be a composition of valuations on K. If (K,w) and $(Kw,\bar{w})$ are defectless, then so is (K,v) [Kuh10a, Lemma 2.17].

(D5) Let $(F/K,v)$ be a valued function field without transcendence defect, and let $v=w\circ\bar{w}$. Then $(F/K,w)$ and $(Fw/Kw,\bar{w})$ are without transcendence defect, and Fw/Kw is finitely generated. To see this, we apply the Abhyankar Inequality to w and to $\bar{w}$. Both of these are in fact equalities, since their sum is the Abhyankar Inequality for v, which is an equality by assumption. For finite generation, we rely on [Bou89, VI §10.3].

Lemma 2.27. *Suppose that* (H) *holds and that (K,v) is defectless. If $(F/K,v)$ is strongly inertially generated, then every normal extension E/F of degree p is defectless.*

Proof. If E/F is Galois, then $(F/K,v)$ admits a strong basis by Proposition 2.14, and E/F has a simplified minimal polynomial by Proposition 2.22. Since K is defectless, case (RT-BASE) of Proposition 2.24 cannot occur, and in each of the other four cases E/F is defectless. Otherwise E/F is purely inseparable of degree p, so v has equal characteristic p. Let x be as in the definition of strongly inertially generated. Since $F/K(x)$ is separably algebraic and K is perfect, $F=F^p(x)$, so $[F:F^p]=p$ and $E=F^{1/p}$. Hence $E=F(y^{1/p})$ for every $y\in F\setminus F^p$. In case (VT), write $\Gamma_F=\Gamma_K\oplus\mathbb{Z}\gamma$ (Remark 2.2**(iii)**) and take $y\in F$ with $v(y)=\gamma$. As Γ_K is p-divisible, $\gamma\notin p\Gamma_F$, so $y\notin F^p$ and $v(y^{1/p})=\frac{1}{p}\gamma\notin\Gamma_F$. Thus $e(E/F)=p$. In case (RT), k_F is a function field of transcendence degree 1 over the perfect field k_K, so $k_F\neq k_F^p$. Take $y\in\mathcal{O}_F^\times$ with $\overline{y}\notin k_F^p$. Then $y\notin F^p$ and $\overline{y^{1/p}}=\overline{y}^{1/p}\notin k_F$. Thus $f(E/F)=p$. In both cases $d(E/F)=1$. □

In Lemma 2.28 and Proposition 2.30 we work under the following hypothesis.

(H$_{\mathrm{AC}}$) K is algebraically closed, and $(F/K,v)$ is a henselized function field of transcendence degree 1 and rank 1, without transcendence defect, and of residue characteristic $p>0$.

We observe that (H$_{\mathrm{AC}}$) implies (H): K is henselian, closed under p-th roots and contains all roots of unity, and $(F/K,v)$ is valuation regular since Γ_K is divisible and k_K is algebraically closed. Moreover (H$_{\mathrm{AC}}$) is inherited by every finite extension of F in place of F, and by every henselized function field $F_0\subseteq F$ over K of transcendence degree 1.

Lemma 2.28. *Assume* (H$_{\mathrm{AC}}$). *Suppose that $(F/K,v)$ is strongly inertially generated, and let N/F be a finite extension of degree prime to p. Then $(N/K,v)$ is strongly inertially generated.*

Proof. Let x be as in the definition of strongly inertially generated. Since $[N:F]$ is prime to p, we have $d(N/F)=1$, so $[N:F]=e(N/F)f(N/F)$. In case (VT), $k_N=k_F=k_K$, as k_K is algebraically closed. So the inert extension $F/K(x)^{\mathrm{h}}$ is trivial, and $n:=(\Gamma_N:\Gamma_F)=[N:F]$. As $\Gamma_{K(x)}=\Gamma_F\neq\Gamma_K$, and every polynomial over the algebraically closed field K is a product of linear ones, we have $v(x-c_0)\notin\Gamma_K$ for some $c_0\in K$.

Replacing x by $x - c_0$, we get $\Gamma_F = \Gamma_K \oplus \mathbb{Z}v(x)$ [Bou89, VI §10.3, Theorem 1], and $\Gamma_N = \Gamma_K \oplus \mathbb{Z}\frac{1}{n}v(x)$ since Γ_K is divisible. Choose $z \in N$ with $v(z) = \frac{1}{n}v(x)$. Then $v(z^n x^{-1}) = 0$ and its residue lies in k_K, so $z^n x^{-1} = cu$ with $c \in K$ and $u \in 1 + \mathfrak{m}_N$, and $u = u_1^n$ for some $u_1 \in N$ by Lemma 1.4**(ii)**. Put $y := zu_1^{-1}$, so that $y^n = cx$. Then $K(x)^{\mathrm{h}} \subseteq K(y)^{\mathrm{h}} \subseteq N$ and $e(K(y)^{\mathrm{h}}/K(x)^{\mathrm{h}}) \geq n = [N : K(x)^{\mathrm{h}}]$, so $N = K(y)^{\mathrm{h}}$, and y is value transcendental over K. In case (RT), $\Gamma_F = \Gamma_K$ is divisible, so $\Gamma_N = \Gamma_F$. Hence $[N : F] = f(N/F)$, which is prime to p. So k_N/k_F is separable, N/F is inert, and hence so is $N/K(x)^{\mathrm{h}}$. □

Remark 2.29. Lemma 2.28 fails if the hypothesis that K is algebraically closed is weakened to the hypothesis that (K, v) is tame. In case (RT), the proof uses only that Γ_K is divisible, whereas the value group of a tame field need only be p-divisible. The following example, adapted from [ADJ24], shows that this is a genuine obstruction. On the other hand, adding to tameness that Γ_K is divisible and k_K is algebraically closed gives nothing new, since a tame field with these properties is algebraically closed.

Let p be an odd prime number and denote $K := \overline{\mathbb{F}}_p(\!(t^{\mathbb{Z}[1/p]})\!)$, which is tame, with $v(t) = 1$. Let $F := K(x)^{\mathrm{h}}$, where $v(x) = 0$ and $\overline{x}$ is transcendental over k_K, so that $(F/K, v)$ is strongly inertially generated and in case (RT). Next we consider $N := F(\sqrt{tx})$. Since $v(\sqrt{tx}) = \frac{1}{2} \notin \Gamma_F = \mathbb{Z}[1/p]$, we have $e(N/F) = 2$, and $[N : F] = 2$ is prime to p. However $|\Gamma_N/\Gamma_K| = 2$, so no $y \in N$ is value transcendental over K. If $y \in N$ is residue transcendental, then $\Gamma_{K(y)^{\mathrm{h}}} = \Gamma_K \neq \Gamma_N$, so $N/K(y)^{\mathrm{h}}$ is not inert. Thus $(N/K, v)$ is not strongly inertially generated. Nor does it help to replace K by its relative algebraic closure in N, which is K itself, since if $K(\sqrt{a}) \subseteq N$ with $a \in K \setminus K^{\times 2}$, then $N = F(\sqrt{a})$, as F/K is regular, so $tx/a \in F^{\times 2}$. Writing $a = ts^2u$ with $s \in K^\times$ and $u \in \mathcal{O}_K^\times$, we would get $\overline{x}/\overline{u} \in k_F^{\times 2}$, which is impossible in $k_F = \overline{\mathbb{F}}_p(\overline{x})$.

Proposition 2.30. *Assume* ($\mathbf{H_{AC}}$). *Then* $(F/K, v)$ *is strongly inertially generated and* (F, v) *is defectless.*

Proof. We first find a strongly inertially generated field $F_0 \supseteq K$, satisfying ($\mathbf{H_{AC}}$) in place of F, such that F/F_0 is finite and immediate. By (H) and Remark 2.2**(iii)**, case (VT) or case (RT) holds. In case (VT), $k_F = k_K$ and $\Gamma_F = \Gamma_K \oplus \mathbb{Z}\gamma$. Choose $x \in F$ with $v(x) = \gamma$, and put $F_0 := K(x)^{\mathrm{h}}$. In case (RT), $\Gamma_F = \Gamma_K$ and k_F/k_K is a function field of transcendence degree 1. As k_K is algebraically closed, $k_F = k_K(\xi, \eta)$ with ξ transcendental and η separably algebraic over $k_K(\xi)$. Choose $x \in F$ with $\overline{x} = \xi$, and, by Hensel's Lemma, $y \in F$ with $\overline{y} = \eta$ and $[K(x)^{\mathrm{h}}(y) : K(x)^{\mathrm{h}}] = [k_K(\xi, \eta) : k_K(\xi)]$. Put $F_0 := K(x)^{\mathrm{h}}(y)$. In both cases F_0 is strongly inertially generated, and F/F_0 is finite and immediate.

Now let F'/F be a finite immediate extension, allowing $F' = F$. Then F'/F_0 is finite and immediate. By **(D3)** there is a finite extension N/F_0 of degree prime to p such that $F'N/N$ is a tower $N = N_0 \subset \cdots \subset N_m = F'N$ of normal extensions of degree p and $d(F'N/N) = d(F'/F_0) = [F' : F_0] \geq [F'N : N]$, so that $F'N/N$ is immediate. If this tower is nontrivial, i.e. $m \geq 1$, then N is strongly inertially generated by Lemma 2.28, so the immediate extension N_1/N of degree p is defectless by Lemma 2.27, which is absurd. Hence $F' \subseteq N$, and $[F' : F_0] = d(F'/F_0)$ is a power of p dividing $[N : F_0]$, so $F' = F_0$. As $F_0 \subseteq F \subseteq F'$, we get $F' = F = F_0$. So $(F/K, v)$ is strongly inertially generated, and F is algebraically maximal.

Finally, every finite extension of F satisfies ($\mathbf{H_{AC}}$) in place of F, so it is algebraically maximal by what we have just shown. By **(D3)**, (F, v) is defectless. □

Proof of Theorem 1. We show that if (K, v) is defectless and $(F/K, v)$ is a valued function field without transcendence defect, then (F, v) is defectless. We proceed by three reductions.

Step 1: reduction to $K(x)$**.** Let $r = \mathrm{rrk}(\Gamma_F/\Gamma_K)$ and $s = \mathrm{trdeg}(k_F/k_K)$, so that $r + s = \mathrm{trdeg}(F/K)$. Choose $t_1, \ldots, t_r \in F$ whose values are rationally independent modulo Γ_K, and $u_1, \ldots, u_s \in F$ of value 0 whose residues are algebraically independent over k_K. By Lemma 1.2, $T := \{t_1, \ldots, t_r, u_1, \ldots, u_s\}$ is algebraically independent over K, and the value of a polynomial in the elements of T is the minimum of the values of its monomials. Hence, for every $T' \subseteq T$, the value group of $K(T')$ is $\Gamma_K \oplus \bigoplus_{t_i \in T'} \mathbb{Z}v(t_i)$ and its residue field is $k_K(\overline{u_j} \mid u_j \in T')$. So T is a transcendence basis of F/K, and $F/K(T)$ is finite. By **(D2)** it suffices to show that $K(T)$ is defectless. Moreover, each $x \in T$ is valuation transcendental over $K(T \setminus \{x\})$. By induction on $|T|$, it therefore suffices to prove:

(†) If (K, v) is defectless and x is valuation transcendental over K, then $(K(x), v)$ is defectless.

Step 2: reduction to K **algebraically closed of finite rank.** Suppose (†) holds for algebraically closed base fields of finite rank, and let (K, v) be defectless and x valuation transcendental over K. Let $E = K(x)^{\mathrm{h}}(a_1, \ldots, a_n)$ be a finite extension. The a_i are algebraic over $K(x)$, so there is a subfield $k_1 \subseteq K$, finitely generated over the prime field, with $a_1, \ldots, a_n$ algebraic over $k_1(x)$. Let k be the algebraic closure of k_1 in K^{alg}. It is algebraically closed, and of finite rank, since $\mathrm{rrk}(\Gamma_{k_1})$ is finite by the Abhyankar Inequality and Γ_k/Γ_{k_1} is torsion. Put $M := K^{\mathrm{alg}}(x)^{\mathrm{h}}$ and $E_k := k(x)^{\mathrm{h}}(a_1, \ldots, a_n)$, so that $EM = E_kM$. The element x is valuation transcendental over k and over K^{alg}, so $d(E_k/k(x)^{\mathrm{h}}) = 1$ by hypothesis and **(D1)**. The extension $(M/k(x)^{\mathrm{h}}, v)$ is valuation

regular, since Γ_k is divisible and k_k is algebraically closed. So Lemma 2.1 gives $d(EM/M) \le d(E_k/k(x)^{\mathrm{h}}) = 1$. Now M is the directed union of the fields $M_L := L(x)^{\mathrm{h}} = LK(x)^{\mathrm{h}}$ over finite subextensions L/K of K^{alg}/K (Lemma 3.17**(i)**). Choose L so large that $[EM_L : M_L] = [EM : M]$, and that EM_L contains representatives of Γ_{EM}/Γ_M and elements whose residues form a basis of k_{EM}/k_M. Then $e(EM_L/M_L) \ge e(EM/M)$ and $f(EM_L/M_L) \ge f(EM/M)$, so $d(EM_L/M_L) \le d(EM/M) = 1$. The extension $(K(x)^{\mathrm{h}}/K^{\mathrm{h}}, v)$ is valuation regular, since x is valuation transcendental over K. So Lemma 2.1 and **(D1)** give $d(M_L/K(x)^{\mathrm{h}}) \le d(LK^{\mathrm{h}}/K^{\mathrm{h}}) = 1$. By multiplicativity, $d(EM_L/K(x)^{\mathrm{h}}) = 1$, and so $d(E/K(x)^{\mathrm{h}}) = 1$, as it divides the former.

Step 3: reduction to rank 1. Let K be algebraically closed of finite rank, and x valuation transcendental over K. Then v has finite rank on $F := K(x)$, and we show by induction on this rank that (F, v) is defectless. If v is trivial on F, there is nothing to prove, since every finite extension of a trivially valued field has a unique, trivial, prolongation with $e = 1$ and f equal to the degree. Otherwise write $v = w \circ \bar{w}$ with w of rank 1, where $\bar{w}$ is trivial if v has rank 1. By **(D5)**, $(F/K, w)$ is a function field of transcendence degree 1 without transcendence defect over the algebraically closed field K. So (F, w) is defectless by Proposition 2.30 and **(D1)**, or by Ostrowski's lemma if w has residue characteristic 0. Also by **(D5)**, $(Fw/Kw, \bar{w})$ is a finitely generated extension without transcendence defect of the algebraically closed field Kw, of transcendence degree at most 1, and $\bar{w}$ has smaller rank than v. If Fw/Kw is algebraic, then $Fw = Kw$ is algebraically closed, hence defectless. Otherwise Fw is finite over $Kw(x')$ for some x' valuation transcendental over $(Kw, \bar{w})$, so $(Fw, \bar{w})$ is defectless by induction and **(D2)**. By **(D4)**, (F, v) is defectless. □

Proposition 2.31 (Strong inertial generation, cf. [KK05, Theorem 3.4], [Kuh16, Theorem 1.9])**.** *Let (K, v) be defectless, and let $(F/K, v)$ be a henselized function field without transcendence defect such that k_F/k_K is separably generated. Then Γ_F/Γ_K and k_F/k_K are finitely generated, and the following hold.*

- **(i)** *$(F/K, v)$ is strongly inertially generated if and only if Γ_F/Γ_K is torsion-free.*
- **(ii)** *Suppose that Γ_F/Γ_K is torsion-free, hence free. Then $F/K(t, u)^{\mathrm{h}}$ is inert for all tuples $t = (t_1, \dots, t_r)$ and $u = (u_1, \dots, u_s)$ in F such that $v(t_1), \dots, v(t_r)$ is a basis of Γ_F/Γ_K, $v(u_j) = 0$ for all j, and $\overline{u_1}, \dots, \overline{u_s}$ is a separating transcendence basis of k_F/k_K. Such tuples exist.*
- **(iii)** *If (H) holds, then $(F/K, v)$ is strongly inertially generated.*

Proof. Put $r := \mathrm{rrk}(\Gamma_F/\Gamma_K)$ and $s := \mathrm{trdeg}(k_F/k_K)$, so that $\mathrm{trdeg}(F/K) = r + s$. First let t and u be any tuples in F as in Lemma 1.2, of lengths r and s. They exist, since Γ_F/Γ_K has rational rank r and k_F/k_K has transcendence degree s. By Lemma 1.2, their entries form a transcendence basis of F/K. Write $F = E^{\mathrm{h}}$ with E/K finitely generated. Then $E(t, u)/K(t, u)$ is finite and $F = E(t, u)K(t, u)^{\mathrm{h}}$, so $F/K(t, u)^{\mathrm{h}}$ is finite. Hence $\Gamma_F/\Gamma_{K(t,u)}$ and $k_F/k_{K(t,u)}$ are finite. By Lemma 1.2, $\Gamma_{K(t,u)}/\Gamma_K$ and $k_{K(t,u)}/k_K$ are finitely generated. So Γ_F/Γ_K and k_F/k_K are finitely generated.

For **(ii)**, Γ_F/Γ_K is free of rank r, being finitely generated and torsion-free. So tuples as in **(ii)** exist: lift a basis of Γ_F/Γ_K to elements t_i, and a separating transcendence basis of k_F/k_K to units u_j. Now let t and u be as in **(ii)**. Everything shown above holds for them. Moreover $\Gamma_{K(t,u)} = \Gamma_K \oplus \bigoplus_i \mathbb{Z}v(t_i) = \Gamma_F$, and $k_F/k_{K(t,u)}$ is finite and separable, since $k_{K(t,u)} = k_K(\overline{u})$. The extension $(K(t, u)/K, v)$ is a valued function field without transcendence defect, by Lemma 1.2. By Theorem 1 and **(D1)**, $K(t, u)^{\mathrm{h}}$ is defectless. Therefore $[F : K(t, u)^{\mathrm{h}}] = e\,f = [k_F : k_{K(t,u)}]$, i.e. $F/K(t, u)^{\mathrm{h}}$ is inert.

For **(i)**, if $(F/K, v)$ is strongly inertially generated with t and u as in the definition, then $\Gamma_F/\Gamma_K \cong \mathbb{Z}^r$ is torsion-free. The converse is **(ii)**. For **(iii)**, under (H) the extension $(F/K, v)$ is valuation regular, so Γ_F/Γ_K is torsion-free, and **(i)** applies. □

If k_K is perfect, for instance if (K, v) is tame, then k_F/k_K is separably generated, being finitely generated. Thus, a posteriori, the hypothesis that $(F/K, v)$ is strongly inertially generated in Proposition 2.14 and Lemma 2.27 is automatic when (K, v) is defectless. This cannot be used to shorten the proof of Theorem 1, on which Proposition 2.31 rests.

2.6. **A variant for separable defectlessness.** The following variant of Theorem 1 is the last assertion of [Kuh10a, Theorem 1.1], and the proof below follows Kuhlmann's. It is needed only in Section 3. It uses **(D1)** together with one further standard fact:

- **(D6)** A henselian field of characteristic p is separably defectless if and only if its completion is defectless [Kuh10b, Theorem 5.2].

Corollary 2.32 (Separably defectless variant, [Kuh10a, Theorem 1.1])**.** *Let (K, v) be separably defectless, and let $(F/K, v)$ be a valued function field without transcendence defect such that Γ_K is cofinal in Γ_F. Then (F, v) is separably defectless.*

Proof. If K has characteristic 0 then every finite extension is separable, and this is Theorem 1. So suppose that K has characteristic p. We use **(D1)** and **(D6)**, and the fact that the completion of a henselian field is henselian [War89, Theorem 32.19].

By **(D1)**, K^{h} is separably defectless, so by **(D6)** its completion $K' := (K^{\mathrm{h}})^c$ is defectless. Since Γ_K is cofinal in Γ_F, the topology of K^{h} is the one induced from F^{h}, so K' embeds over K^{h} into the completion $(F^{\mathrm{h}})^c$. Let $F' := FK' \subseteq (F^{\mathrm{h}})^c$. This is a function field over K', with $\mathrm{trdeg}(F'/K') \leq \mathrm{trdeg}(F/K)$. As $(F^{\mathrm{h}})^c/F$ and K'/K are immediate, we have $\Gamma_{F'} = \Gamma_F$, $k_{F'} = k_F$, $\Gamma_{K'} = \Gamma_K$ and $k_{K'} = k_K$. So the Abhyankar Inequality for F'/K', together with $\mathrm{trdeg}(F/K) = \mathrm{rrk}(\Gamma_F/\Gamma_K) + \mathrm{trdeg}(k_F/k_K)$, shows that F'/K' is without transcendence defect. By Theorem 1, (F', v) is defectless, hence so is (F'^{h}, v), by **(D1)**. In particular F'^{h} is separably defectless. Now $F^{\mathrm{h}} \subseteq F'^{\mathrm{h}} \subseteq (F^{\mathrm{h}})^c$, the latter being henselian, so the completion of F'^{h} is $(F^{\mathrm{h}})^c$, which is therefore defectless by **(D6)**. By **(D6)** again, F^{h} is separably defectless, and by **(D1)**, so is F. □

3. Henselian rationality

We now treat the case complementary to Section 2: that of immediate henselized function fields, i.e. pure transcendence defect. The standing hypothesis is the following analogue of (H).

(H$_{\mathrm{IMM}}$) Let (K, v) be a henselian, separably algebraically maximal valued field of residue characteristic $p > 0$ and of rank 1. If $\mathrm{char}K = p$, suppose that (K, v) is dense in (K^{perf}, v), and if $\mathrm{char}K = 0$, suppose that K is closed under p-th roots and contains all p-th roots of unity. Let $(F/K, v)$ be an immediate separable henselized function field of transcendence degree 1, say $F = E^{\mathrm{h}}$ with E/K finitely generated.

We refer to this situation, in which $(F/K, v)$ is immediate, as case (IMM), complementing the cases (VT) and (RT) of Section 2.

Remark 3.1. From (H$_{\mathrm{IMM}}$) it follows that

- **(i)** $\Gamma_F = \Gamma_K$ is p-divisible and $k_F = k_K$ is perfect: in mixed characteristic as in Remark 2.2**(i)**,**(ii)** together with immediacy, and in equal characteristic because density of K in K^{perf} gives $\Gamma_K = \Gamma_{K^{\mathrm{perf}}}$ and $k_K = k_{K^{\mathrm{perf}}}$. The hypotheses on K hold whenever K is closed under p-th roots, and in equal characteristic whenever (K, v) is separably tame, by Corollary 1.13.
- **(ii)** ("Transcendental approximation type") For every $y \in F$ transcendental over K and every $h \in K[Y]$ there is $\alpha \in v(y-K)$ such that $vh(c)$ takes a constant value, which we denote by $v_y h$, for all $c \in K$ with $v(y-c) \geq \alpha$. We say that h **stabilizes** at α. This is where the hypothesis that (K, v) is separably algebraically maximal is used (Lemma 1.14**(ii)**, applied to $K(y) \subseteq F$, which also shows that $v_y h = v(h(y))$). It will be used only in Lemma 3.5. The rigidity Lemma 3.8 does not need it.

Finally we fix the terminology of the section, which was left informal in Theorem 2.

Definition 3.2. An extension $(E/K, v)$ of valued fields is **henselian rational** if there is a transcendence basis $T \subseteq E$ of E/K such that $E \subseteq K(T)^{\mathrm{h}}$. Equivalently, $E^{\mathrm{h}} = K(T)^{\mathrm{h}}$. Note that T is required to lie in E itself, and not merely in its henselization: this is what makes the property usable in the reduction to finite rank, Proposition 3.22.

3.1. **Frames.** The following observation explains why the strong bases of Section 2 do not exist in this context, and why (SB-I) and (SB-IV) must be replaced: the RV-structure is blind to immediate extensions.

Lemma 3.3. *Let $(F/K, v)$ be immediate. Then the natural map $\mathrm{RV}_K \to \mathrm{RV}_F$ is an isomorphism. If moreover Γ_K is p-divisible and k_K is perfect, then $\mathrm{RV}_F = \mathrm{RV}_F^p$. Consequently, under (H$_{\mathrm{IMM}}$), no $\mathcal{B} \subseteq F^\times$ with $1 \in \mathcal{B}$ and $\mathcal{B} \neq \{1\}$ satisfies (SB-I), and (SB-IV) is vacuous, so it carries no information in this setting.*

Proof. Injectivity holds since $K^\times \cap (1 + \mathfrak{m}_F) = 1 + \mathfrak{m}_K$. For surjectivity, let $a \in F^\times$: by immediacy there is $t \in K^\times$ with $v(ta) = 0$ and $e \in \mathcal{O}_K^\times$ with $\overline{e} = \overline{ta}$. Then $a(t^{-1}e)^{-1} \in 1 + \mathfrak{m}_F$, so $\mathrm{rv}(a) = \mathrm{rv}(t^{-1}e)$. The equality $\mathrm{RV}_F = \mathrm{RV}_F^p$ follows from Lemma 2.5**(iii)**, since every value lies in $p\Gamma_F = \Gamma_F$. In particular the hypothesis of (SB-IV) holds for every sum. For the failure of (SB-I): given $b \in \mathcal{B} \setminus \{1\}$, choose t, e as above. Then $v(tb - e) > 0 = \min\{v(tb), v(e)\}$, contradicting (SB-I) for $B = \{1, b\}$ with coefficients $a_b = t$, $a_1 = -e$. □

Definition 3.4. Fix $x \in F$ transcendental over K. For $c \in K$ we choose, using immediacy, some $d_c \in K^\times$ with $vd_c = v(x - c)$, and we set

$$z_c := \frac{x - c}{d_c}, \qquad \mathcal{B}_c := \{z_c^i \mid i \in \mathbb{N}\}.$$

Then $vz_c = 0$ and $K\langle \mathcal{B}_c \rangle = K[\mathcal{B}_c] = K[x]$ for every c, since $\mathcal{B}_c$ is closed under products. The family $(\mathcal{B}_c)_c$ is directed by $v(x - c)$, and the change of frame $\mathcal{B}_c \to \mathcal{B}_{c'}$ is the affine Taylor substitution. For $c' \in K$ we

have $v(z_c - c') = v\big(x - (c + d_c c')\big) - vd_c$, so the substitution $x \mapsto z_c$ carries $v(x-K)$ onto $v(z_c - K)$ by the translation $\gamma \mapsto \gamma - vd_c$. Hence each $(K(z_c)/K, v)$ is again immediate and transcendental, with transcendental approximation type (see Observation 1.1 and Remark 3.1**(ii)**).

Throughout, for $f \in K[X]$ we write $f(X) = \sum_{i=0}^{n} f_i(X_0)(X - X_0)^i$ for the Taylor expansion with Hasse–Schmidt derivatives $f_i \in K[X_0]$, where $n = \deg f$.

Lemma 3.5 (Taylor separation)**.** *Assume* ($\mathbf{H_{IMM}}$) *and let $f \in K[x]$ be of degree n. There is $\alpha \in v(x-K)$ such that for all $c \in K$ with $v(x-c) \geq \alpha$:*

- **(i)** *$f_i(c) \neq 0$ whenever $f_i \neq 0$, and the values $vf_i(c)$ are constant, equal to $\gamma_i := v_x f_i$;*
- **(ii)** *$vf(x) = vf(c) < v\big(f_i(c)(x-c)^i\big)$ for all $1 \leq i \leq n$ with $f_i \neq 0$.*

Moreover, c with $v(x-c) \geq \alpha$ may be chosen such that

- **(iii)** *the values $v\big(f_i(c)(x-c)^i\big)$, $1 \leq i \leq n$ with $f_i \neq 0$, are pairwise distinct (and, if desired, avoid any prescribed finite subset of Γ_K).*

Proof. By Remark 3.1**(ii)** choose $\alpha_0 \in v(x-K)$ at which f and each nonzero f_i stabilize. Since a nonzero f_i vanishes at only finitely many points, stabilization forces $f_i(c) \neq 0$ for $v(x-c) \geq \alpha_0$, giving **(i)**. For $\beta := v(x-c) \geq \alpha_0$ and $i \geq 1$ we have $v(f_i(c)(x-c)^i) = \gamma_i + i\beta$, an affine function of β with slope i. Any two of these agree for at most one value of β. Call β *separated* if all these values are pairwise distinct. By Observation 1.1 all but finitely many levels $\beta \in v(x-K)$, $\beta \geq \alpha_0$, are separated, and they are cofinal in $v(x-K)$. At a separated level realized by c,

$$v\big(f(x) - f(c)\big) = m(\beta) := \min_{1 \leq i \leq n} \big(\gamma_i + i\beta\big),$$

and m is strictly increasing in β. If $m(\beta) < \gamma_0$ at two separated levels $\beta < \beta'$, then $vf(x) = m(\beta) = m(\beta')$, contradicting strict increase, and $m(\beta) = \gamma_0$ at most once. So some separated level α satisfies $m(\alpha) > \gamma_0$, and then $m(\beta) > \gamma_0$ for all $\beta \geq \alpha$ since m is increasing. As $v(f(x) - f(c)) \geq m(\beta)$ always, **(ii)** holds at every such level. **(iii)** is the choice of a separated level avoiding finitely many further collisions. □

Lemma 3.6 (Span density, (SB-II))**.** *Assume* ($\mathbf{H_{IMM}}$)*. Then $K[x] = K\langle \mathcal{B}_c \rangle$ is dense in $(K(x)^{\mathrm{h}}, v)$.*

Proof. Since v is of rank 1, completions are henselian and $K(x)$ is dense in $K(x)^{\mathrm{h}}$, as in the proof of Proposition 2.14. It remains to approximate $1/f(x)$ for $0 \neq f \in K[x]$. Choose c as in Lemma 3.5 and set $u := 1 - f(c)^{-1} f(x) \in K[x]$, so that $vu = v(f(x) - f(c)) - vf(c) > 0$. Then

$$\frac{1}{f(x)} - f(c)^{-1} \sum_{k=0}^{N} u^k = f(c)^{-1} \frac{u^{N+1}}{1-u},$$

of value $-vf(c) + (N+1)vu$, which is cofinal in Γ_F as $N \to \infty$ since Γ_F is archimedean. This is [KV14, Lemma 10.1]. Note it is the same geometric-series-plus-rank-1 argument as in Proposition 2.14, with the 1-unit $1 - u$ produced by Taylor separation instead of valuation independence. □

Remark 3.7 (Frames versus strong bases)**.** The family $(\mathcal{B}_c)_c$ plays the role here that a strong basis plays in Section 2, with two substitutions. Density (SB-II) holds by Lemma 3.6, and Frobenius closure (SB-III) holds since $(z_c^i)^p = z_c^{pi} \in \mathcal{B}_c$, the surgery outputs $(az_c^{pi})^{1/p} = a^{1/p} z_c^i$ and $-p(az_c^{pi})^{1/p} = -pa^{1/p} z_c^i$ lying in $K^\times \cdot \mathcal{B}_c$ in mixed characteristic, where K is closed under p-th roots, and in $K^{\mathrm{perf}} \cdot \mathcal{B}_c$ in equal characteristic, whence back in $K^\times \cdot \mathcal{B}_c$ after the approximation performed in Proposition 3.10. This is where the purely inseparable extension K' of [Kuh19a, Lemma 4.2] is eliminated. By contrast, valuation independence (SB-I) and multiplicative rigidity (SB-IV) are both unavailable (Lemma 3.3): the first is replaced by the eventual separation of Lemma 3.5, the second by the rigidity Lemma 3.8. Termination of the surgery, which in Section 2 required (SB-IV) by way of Lemma 2.17, is here immediate, since the exponent pj drops to j.

3.2. Normal forms and the Galois p extensions.

Lemma 3.8 (Ershov)**.** *Let (K, v) be a valued field of residue characteristic $p > 0$, and let $(K(z)/K, v)$ be immediate with $z \notin K$ and $v(z) = 0$. Let $f(X) = \sum_{i=0}^{n} a_i X^i \in K[X]$ and suppose that, for some $i_0 \in \{1, \dots, n\}$ with $p \nmid i_0$, we have $va_{i_0} < va_i$ for all $i \in \{1, \dots, n\} \setminus \{i_0\}$. Then*

$$K(z)^{\mathrm{h}} = K(f(z))^{\mathrm{h}}.$$

Proof. Since $f(z) \in K(z)$ we have $K(f(z))^{\mathrm{h}} \subseteq K(z)^{\mathrm{h}}$, so only $z \in K(f(z))^{\mathrm{h}}$ needs proof.

By Observation 1.1 the set $v(z-K)$ has no maximal element. Taking $c = 0$ shows $0 = v(z) \in v(z-K)$, so we may fix $c \in K$ with $v(z-c) > 0$. Then $v(c) = v\big(z - (z-c)\big) = v(z) = 0$.

Write $f(z)-f(c)=\sum_{k\geq 1} f_k(c)(z-c)^k$ with $f_k(c)=\sum_{i\geq k}\binom{i}{k}a_i c^{i-k}$. Since $v(c)=0$ and $v\binom{i}{k}\geq 0$,

$$v(f_k(c))\geq \min_{i\geq k} v(a_i)\geq v(a_{i_0}) \qquad (k\geq 1).$$

For $k=1$ equality holds: in $f_1(c)=\sum_{i\geq 1} i a_i c^{i-1}$ the i_0-th summand has value $v(i_0)+v(a_{i_0})=v(a_{i_0})$, because $p\nmid i_0$, while every other summand has value $\geq v(a_i)>v(a_{i_0})$. The minimum is thus attained uniquely and $v(f_1(c))=v(a_{i_0})$. In particular $f_1(c)\neq 0$. Hence for $k\geq 2$,

$$v\big(f_k(c)(z-c)^k\big)\geq v(a_{i_0})+k\,v(z-c)>v(a_{i_0})+v(z-c)=v\big(f_1(c)(z-c)\big),$$

using $v(z-c)>0$, so the summand with $k=1$ strictly dominates.

Put $h:=f(z)-f(c)$, so that $K(h)=K(f(z))$. By immediacy $v(z-c)\in\Gamma_K$. Choose $b\in K$ with $v(b)=v(z-c)$ and, after multiplying b by a unit of K whose residue is $\overline{(z-c)/b}$ (possible as $k_{K(z)}=k_K$), set

$$y:=\frac{z-c}{b}, \qquad v(y)=0, \qquad \overline{y}=1.$$

Let $h':=h/(f_1(c)b)$ and $\alpha_k:=f_k(c)b^k/(f_1(c)b)$ for $k\geq 2$. Then $v(\alpha_k)=v(f_k(c)(z-c)^k)-v(f_1(c)(z-c))>0$, and dividing the above expansion by $f_1(c)b$ gives $h'=y+\sum_{k\geq 2}\alpha_k y^k$, whence $v(h')=0$ and $\overline{h'}=\overline{y}=1$. Consider

$$H(Y):=Y+\sum_{k\geq 2}\alpha_k Y^k-h'\in K(h)[Y].$$

Its coefficients lie in $\mathcal{O}_{K(h)}$, we have $H(y)=0$, and $\overline{H}(Y)=Y-1$, which has 1 as a simple root. Hensel's Lemma in $K(h)^{\mathrm{h}}$ yields a root $\tilde{y}\in K(h)^{\mathrm{h}}$ with $\overline{\tilde{y}}=1$. Taking all henselizations inside $K(z)^{\mathrm{h}}$, both y and $\tilde{y}$ are roots of H with residue 1 in the henselian field $K(z)^{\mathrm{h}}$, so $y=\tilde{y}\in K(h)^{\mathrm{h}}$ by the uniqueness assertion in Hensel's Lemma.

$z=c+by\in K(h)^{\mathrm{h}}=K(f(z))^{\mathrm{h}}$, so $K(z)\subseteq K(f(z))^{\mathrm{h}}$ and therefore $K(z)^{\mathrm{h}}\subseteq K(f(z))^{\mathrm{h}}$, the latter field being henselian. □

We will also use the following inequality between the monomials of a Taylor expansion at deep centres. It is [Kap42, Lemma 7], see also [KV14, Proposition 7.4] and [Kuh19a, (4.15)].

Lemma 3.9 (Block domination). *Assume* ($\mathbf{H_{IMM}}$) *and let* $f\in K[x]$. *For sufficiently large* $\alpha\in v(x-K)$ *and all* c *with* $v(x-c)\geq\alpha$: *if* $j=p^t$ *and* $i=p^t r$ *with* $t\geq 0$, $r>1$, $(r,p)=1$, *then* $vf_j(c)(x-c)^j<vf_i(c)(x-c)^i$, *unless both are infinite.*

Proposition 3.10 (Simplified minimal polynomials, case (IMM)). *Assume* ($\mathbf{H_{IMM}}$), *let* $F=K(x)^{\mathrm{h}}$, *and let* $(E/F,v)$ *be a Galois extension of degree* p. *Then there is a frame* $z=z_c$ *such that:*

(EQ) *If* v *has equal characteristic, then* E/F *has a minimal polynomial*

$$g=Z^p-Z-a, \qquad a=a_0+\sum_{i\in I}a_i z^i,$$

with $a_0\in K$, $I\subseteq\mathbb{N}_{\geq 1}$ *finite,* $a_i\in K^\times$, *which is* **simplified**:
(EQ-I′) $v(a_i)<0$ *for all* $i\in I$, *and*
(EQ-II′) $I=\emptyset$, *or there is* $i_0\in I$ *with* $p\nmid i_0$ *such that* $v(a_{i_0})<v(a_i)$ *for all* $i\in I\setminus\{i_0\}$.

(MIX) *If* v *has mixed characteristic, then* E/F *has a minimal polynomial*

$$g=Z^p-a, \qquad a=1+\sum_{i\in I}a_i z^i,$$

with $I\subseteq\mathbb{N}_{\geq 1}$ *finite nonempty and* $a_i\in K^\times$, *which is* **simplified**:
(MIX-I′) $0<v(a_i)\leq\frac{p}{p-1}v(p)$ *for all* $i\in I$, *and*
(MIX-II′) *there is* $i_0\in I$ *with* $p\nmid i_0$ *such that* $v(a_{i_0})<v(a_i)$ *for all* $i\in I\setminus\{i_0\}$.

Proof. We begin in case (EQ). As in Proposition 2.22, E/F has a minimal polynomial $Z^p-Z-\alpha$ with $\alpha\in F$, determined up to $\alpha+\wp(F)$. By Lemma 3.6 and $\mathfrak{m}_F\subseteq\wp(F)$ (Lemma 1.4**(i)**) we may take $\alpha=f(x)$ with $f\in K[x]$ of degree n.

We perform the Frobenius surgery *symbolically*, before choosing the frame. In the Taylor expansion $f(X)=\sum_i f_i(X_0)(X-X_0)^i$, each summand with exponent pj satisfies

$$f_{pj}(X_0)(X-X_0)^{pj}\equiv f_{pj}(X_0)^{1/p}(X-X_0)^j \mod \wp\big(K^{1/p}[X,X_0^{1/p}]\big),$$

the coefficient lying in $K[X_0]^{1/p} \subseteq K^{1/p}[X_0^{1/p}]$. Iterating — the p-adic valuation of each exponent strictly drops, so the process terminates — we obtain

$$f(X) \equiv f(X_0) + \sum_{\substack{1\le j\le n\\ p\nmid j}} r_j(X_0)(X-X_0)^j, \qquad r_j(X_0) = \sum_{\nu\ge 0,\ jp^\nu\le n} f_{jp^\nu}(X_0)^{p^{-\nu}},$$

modulo $\wp\big(K^{\mathrm{perf}}[X, X_0^{1/p^\infty}]\big)$. The r_j need not be polynomials, but $r_j^{p^\lambda} \in K[X_0]$ for large λ, so $v r_j(c) = p^{-\lambda} v\big(r_j^{p^\lambda}(c)\big)$ stabilizes along with polynomial values. Write β_j for the stable value. Now choose the frame: by Lemma 3.5**(iii)** applied to f and the $r_j^{p^\lambda}$, pick c with $v(x-c)$ beyond all stabilization points and such that the finitely many values $\beta_j + j\,v(x-c)$ are pairwise distinct, nonzero, and distinct from $v_x f$. Substituting $X := x$, $X_0 := c$ and writing $b_j := r_j(c) d_c^j$, we get $\alpha \equiv f(c) + \sum_j b_j z_c^j$ modulo $\wp(F^{\mathrm{perf}})$ with $v b_j = \beta_j + j\,v(x-c)$ nonzero and pairwise distinct. Discarding the terms of positive value into $\mathfrak{m}_F \subseteq \wp(F)$ leaves $a = f(c) + \sum_{i\in I} b_i z_c^i$ with $I = \{j \mid v b_j < 0\}$, which satisfies (EQ-I′) and (EQ-II′), the minimum being attained at a unique, prime-to-p, index.

The coefficients b_i and $f(c)$ lie a priori in K^{perf}, so what we have obtained is a congruence modulo $\wp(F')$, where $F' := K^{1/p^\nu}(x)^{\mathrm{h}}$ for a suitable $\nu \in \mathbb{N}$. Since K is dense in K^{perf} we may replace each coefficient by an element of K whose difference from it has value so large that the resulting $a \in K[z_c]$ still satisfies $\alpha - a \in \mathfrak{m}_{F'} \subseteq \wp(F')$. The values $v b_i$ are unchanged, so (EQ-I′) and (EQ-II′) persist. Finally, F'/F is purely inseparable, so $E \mapsto EF'$ is a bijection between the separable extensions of F and those of F'. As $a \in K[z_c] \subseteq F$, the Artin–Schreier extension of F generated by a root of $Z^p - Z - a$ has the same compositum with F' as E does, hence equals E. (If K is closed under p-th roots then $F' = F$ and this last step is vacuous.)If the resulting I is empty, we are in the split case of Proposition 3.12 below.

We turn to case (MIX), following [Kuh19a, Lemma 4.3]. We indicate the steps and where each ingredient enters. First, normalize the Kummer generator: $E = F(\eta)$ with $\eta^p = \alpha \in F^\times \setminus F^{\times p}$, determined up to $\alpha F^{\times p}$. Since $\Gamma_F = \Gamma_K$ is p-divisible and $k_F = k_K$ is perfect (Remark 3.1**(i)**), there are $e_1, e_2 \in K^\times$ with $v e_1^p = -v\alpha$ and $\overline{e_1^p e_2^p \alpha} = 1$. Since $K = K^{\times p} \cup \{0\}$, replacing η by $e_1 e_2 \eta$ replaces α by a 1-unit (this collapses the prefactor x of (MIX)). By Lemma 3.6 choose $f \in K[x]$ with $v(\alpha - (1+f(x))) > \frac{p}{p-1} v p$ and hence, by Lemma 1.4**(iii)**, we may take $\alpha = 1 + f(x)$, $v f(x) > 0$. By Lemma 3.5 choose c_1 deep enough that $v f(x) = v f(c_1) < v f_i(c_1)(x-c_1)^i$ for all $i \ge 1$, and set $y := z_{c_1}$ and $d_i := f_i(c_1) d_{c_1}^i \in \mathfrak{m}_K$, so $f(x) = \sum_i d_i y^i$. Next, kill the p-divisible exponents to first order: choose $s(y) \in K[y]$ with $s(y) \equiv \big(1 + \sum_{p\mid i} d_i^{1/p} y^{i/p}\big)^{-1}$ mod $p\mathfrak{m}_K[y]$ — possible by the geometric series and rank 1, as in Lemma 3.6 — and set

$$\varepsilon(y) := 1 + \sum_{p\mid i} d_i y^i - \Big(1 + \sum_{p\mid i} d_i^{1/p} y^{i/p}\Big)^p \in p\mathfrak{m}_K[y],$$
$$f'(y) := s(y)^p \varepsilon(y) + s(y)^p \sum_{p\nmid i} d_i y^i =: \sum_i d_i' y^i.$$

As $s(y)^p \big(1 + \sum_{p\mid i} d_i^{1/p} y^{i/p}\big)^p \equiv 1$ modulo $p^2\mathfrak{m}_K[y]$, we get $s(y)^p(1+f) \equiv 1 + f'(y)$ modulo $p^2\mathfrak{m}_K[y]$. So by Lemma 1.4**(iii)** we may replace α by $1 + f'(y)$, and then delete from f' every term of value $> \frac{p}{p-1} v p$. Every monomial of f' whose exponent is divisible by p has value $> vp$. Suppose first that some monomial of f' has value at most vp. Such a monomial comes from $s(y)^p \sum_{p\mid i} d_i y^i$, and by Lemma 3.9 the linear term $d_1' y$ is the unique monomial of least value: indeed $d_1' - d_1 = (s_0^p - 1) d_1 + \rho$, where the constant term s_0 of s is a 1-unit and ρ, the coefficient of y in $s(y)^p \varepsilon(y)$, has value $> vp \ge v d_1$. So $v(d_1' - d_1) > v d_1$. Thus (MIX-II′) holds with $i_0 = 1$, and we set $z := y$. Suppose next that every monomial of f' has value $> vp$. As $vy = 0$, this means $f' \in p\mathfrak{m}_K[y]$. In this case we follow Ershov's argument [Eršo8], in the form given in [Kuh19a, pp. 943–945]. Consider pairs (h^*, γ_0) with $\gamma_0 \in v(y - K)$ and $h^*(Y,Z) = \sum_{i=0}^n h_i(Z)(Y-Z)^i \in p\mathfrak{m}_K[Y,Z]$ (here the h_i are coefficients, not Hasse–Schmidt derivatives). Such a pair is **admissible** if, whenever $c \in K$ satisfies $v(y-c) \ge \gamma_0$, **(a)** each value $v h_i(c)$ is independent of c, **(b)** the nonzero terms $h_i(c)(y-c)^i$ have pairwise distinct values, and **(c)** $1 + h^*(y,c) \in \alpha F^{\times p}$. For instance, $\big(\sum_i f_i'(Z)(Y-Z)^i, \gamma_0\big)$ is admissible for γ_0 large enough, by Lemma 3.5**(i)** and since $v(y-K)$ has no largest element (two of the functions $\gamma \mapsto \gamma_i + i\gamma$ agree at most once). Choose an admissible pair for which h^* has as few monomials as possible. Applying the operator $\Delta(u) = -p u^{1/p}$ of Lemma 1.4**(iv)** blockwise moves each exponent ip^m to its prime-to-p root i, with values tracked by $\delta(\gamma) = vp + \frac{1}{p}\gamma$. A collision of δ-levels within a block would allow two monomials to be merged, contradicting minimality — the same cycle-collapse pattern as in the proof of Lemma C.1. Hence the

resulting coefficients $r_i(c)$ have stable, and, for a deep separated choice of c_2, pairwise distinct values. Deleting values $> \frac{p}{p-1}vp$ and setting $z := (y - c_2)/b$ for suitable $b \in K$ yields (MIX-I′) and (MIX-II′), with $z = z_c$ for $c = c_1 + d_{c_1}c_2$. In both cases we have replaced α, modulo $F^{\times p}$, by a 1-unit $1 + \sum_{i\in I\cup\{0\}} a_i z^i$ with $a_i \in K$ and $0 < v(a_i) \leq \frac{p}{p-1}vp$ for all $i \in I \cup \{0\}$, satisfying (MIX-II′). As $1 + a_0 \in K^\times = K^{\times p}$, we may divide by it, and so assume that $a_0 = 0$. This does not change the values $v(a_i)$, since $1 + a_0$ is a 1-unit. So $a = 1 + \sum_{i\in I} a_i z^i$ satisfies (MIX-I′) and (MIX-II′). Finally, $I \neq \emptyset$, since otherwise $a = 1$ and $E = F$. □

Remark 3.11. We briefly compare Propositions 2.22 and 3.10. In contrast to Proposition 2.22(MIX), no prefactor x occurs in Proposition 3.10(MIX). Indeed, by Lemma 3.3 every Kummer generator is congruent to a 1-unit modulo $F^{\times p}$. The conditions correspond to those of Proposition 2.22 as follows. (EQ-I′) and (MIX-I′) correspond to (EQ-II) and (MIX-II), and (EQ-II′) and (MIX-II′) play the role of (EQ-III) and (MIX-III). The condition $I \subseteq \mathbb{N}_{\geq 1}$ in case (MIX) corresponds to (MIX-I). There is no analogue of (EQ-I), since only the exponent i_0 is required to be prime to p, nor of (EQ-IV). Finally, the constant coefficient in case (EQ) is a_1 in Proposition 2.22, indexed by the basis element 1, and a_0 in Proposition 3.10, indexed by the exponent 0.

Proposition 3.12 (Ramification of Galois degree-p extensions, case (IMM))**.** *Continue to assume* ($\mathrm{H_{IMM}}$)*, and let $F = K(x)^{\mathrm{h}}$. Let $(E/F, v)$ be a Galois extension of degree p with simplified minimal polynomial g as in Proposition 3.10, in the frame $z = z_c$: in case (EQ), $g = Z^p - Z - a$ with $a = a_0 + \sum_{i\in I} a_i z^i$, and in case (MIX), $g = Z^p - a$ with $a = 1 + \sum_{i\in I} a_i z^i$. Exactly one of the following two cases holds.*

(SPLIT) *Suppose that (EQ) holds and $I = \emptyset$, i.e. the simplified minimal polynomial g is defined over K, as in case (SPLIT) of Proposition 2.24. Then E is the compositum of F and an Artin–Schreier extension E_0 of K.*

(IMM-HR) *Suppose that $I \neq \emptyset$. Let $\vartheta \in E$ be a root of g, so that $\wp(\vartheta) = a$ in case (EQ), and $\vartheta^p = a$ in case (MIX). Then $E = K(\vartheta)^{\mathrm{h}}$.*

Case (SPLIT) occurs only in equal characteristic: in case (MIX) we have $I \neq \emptyset$, since $K^\times = K^{\times p}$ by ($\mathrm{H_{IMM}}$)*.*

Proof. Case (SPLIT) is as in case (SPLIT) of Proposition 2.24. For case (IMM-HR), set $h(z) := \sum_{i\in I} a_i z^i$ in both cases, so that $h(z) = a - a_0$ in case (EQ) and $h(z) = a - 1$ in case (MIX). By (EQ-II′) resp. (MIX-II′) the polynomial h has a unique coefficient of least value among those of positive degree, at an index prime to p. Moreover, $(K(z)/K, v)$ is immediate with $vz = 0$ and $z \notin K$ (Definition 3.4). Hence Lemma 3.8 yields $K(z)^{\mathrm{h}} = K(h(z))^{\mathrm{h}}$. In case (EQ) we have $h(z) = \wp(\vartheta) - a_0 \in K(\vartheta)$, and in case (MIX) we have $h(z) = \vartheta^p - 1 \in K(\vartheta)$. Therefore

$$F = K(x)^{\mathrm{h}} = K(z)^{\mathrm{h}} = K(h(z))^{\mathrm{h}} \subseteq K(\vartheta)^{\mathrm{h}}.$$

Since $E = F(\vartheta)$ is henselian, it follows that $K(\vartheta)^{\mathrm{h}} \subseteq E = F(\vartheta) \subseteq K(\vartheta)^{\mathrm{h}}$, as required. □

3.3. **The Main Proposition.**

Fact 3.13. *Let (L, v) be separably closed with v nontrivial. Then (L, v) is henselian, Γ_L is divisible, and k_L is algebraically closed.*

Proof. Since $\widetilde{L}/L$ is purely inseparable, v extends uniquely to $\widetilde{L}$, so (L, v) is henselian. For divisibility, let $a \in L^\times$: if $p \nmid n$ then $X^n - a$ is separable and its root witnesses $\frac{1}{n}va \in \Gamma_L$. For $n = p$ we may assume $va < 0$, and a root ϑ of the separable $X^p - X - a$ satisfies $v\vartheta < 0$, whence $pv\vartheta = va$. For k_L: given $\alpha \in k_L$, choose $a \in \mathcal{O}_L$ with $\overline{a} = \alpha$ and $b \in \mathfrak{m}_L \setminus \{0\}$. A root ϑ of $X^p - bX - a$ has $v\vartheta \geq 0$ and $\overline{\vartheta}^p = \alpha$, so k_L is perfect. Moreover, if $g \in \mathcal{O}_L[X]$ is monic with $\overline{g}$ separable, then g splits in L with integral roots, whose residues split $\overline{g}$. Hence k_L is algebraically closed. □

Proposition 3.14 (Completion case, cf. [Kuh19a, Theorem 2.3])**.** *Let (K, v) be henselian of rank 1 and let $(F/K, v)$ be a separable henselized function field of transcendence degree 1, say $F = E^{\mathrm{h}}$, admitting a valuation-preserving K-embedding into (K^c, v). Then $F = K(x)^{\mathrm{h}}$ for every separating element x of E/K. In particular $(E/K, v)$ is henselian rational.*

Proof. Identify $F \subseteq K^c$. Since (K^c, v) is henselian [War89, Theorem 32.19], we have $K(x)^{\mathrm{h}} \subseteq K^c$. The closure of $K(x)^{\mathrm{h}}$ in the complete rank-1 field K^c is a completion of $K(x)^{\mathrm{h}}$, and it contains the closure of K, namely K^c itself, hence contains F. Thus $K(x)^{\mathrm{h}}F/K(x)^{\mathrm{h}}$ is a separably algebraic extension inside the completion of the henselian field $K(x)^{\mathrm{h}}$, and is therefore trivial [War89, Theorem 32.19]. So $F \subseteq K(x)^{\mathrm{h}} \subseteq F$. □

In rank 1 we identify value groups with subgroups of $\mathbb{R}$, and put $\sup v(x - M) := \sup\{v(x - c) \mid c \in M\} \in \mathbb{R} \cup \{\infty\}$.

Lemma 3.15 (Henselian generators in rank 1, cf. [KV14, Theorems 10.7 and 11.1, Corollary 10.8]). *Let (M,v) be separably algebraically maximal of rank 1, let x be transcendental over M with $(M(x)/M,v)$ immediate and $\delta := \sup v(x-M) < \infty$, and put $H := M(x)^{\mathrm{h}}$.*

(i) *If $y \in H$ and $v(y-x) > \delta$, then $H = M(y)^{\mathrm{h}}$.*
(ii) *If $w \in H$ and $H = M(w)^{\mathrm{h}}$, then $v(w-ex-c) > v(e)+\delta$ for some $e \in M^\times$ and $c \in M$.*
(iii) *If $H = E^{\mathrm{h}}$ for a subfield $M \subseteq E \subseteq H$, then $H = M(y)^{\mathrm{h}}$ for some $y \in E$.*

Proof. By Lemma 1.14**(ii)**, for every $f \in M[X]$ we have $v(f(c)) = v(f(x))$ for all $c \in M$ with $v(x-c)$ close enough to δ, briefly: for c near x. As H/M is immediate and separable, and M has no proper immediate separably algebraic extension, every $t \in H \setminus M$ is transcendental over M, and $(M(t)/M,v)$ is immediate. Then $M[t]$ is dense in $M(t)^{\mathrm{h}}$, by the proof of Lemma 3.6, where $f(c)$ is replaced by some $b \in M$ with $v(f(t)-b) > v(f(t))$, given by immediacy. If moreover $M(t)^{\mathrm{h}} = H$, then $\sup v(t-M) < \infty$, as otherwise M would be dense in $M(t)$, hence in H.

Let $f \in M[X]$ be nonconstant, and write $f(X) = \sum_i f_i(x)(X-x)^i$. Let h_f be the largest $i \geq 1$ minimizing $v(f_i(x))+i\delta$, and put $\beta_f := v(f_{h_f}(x))$. For $\gamma < \delta$ close enough to δ, h_f is the only $i \geq 1$ minimizing $v(f_i(x))+i\gamma$. Hence

(1) $v(f(x)-f(c)) = \beta_f + h_f v(x-c)$ for c near x, and
(2) $\sup v(f(x)-M) = \beta_f + h_f\delta$.

For (2), let $b \in M$. Then $v(f(c)-b) = v(f(x)-b)$ for c near x, so $v(f(x)-b) \leq v(f(x)-f(c)) < \beta_f + h_f\delta$.

(3) If $v(f(x)-x) > \delta$, then $h_f = 1$ and $\beta_f = 0$, i.e. $v(f_1(x)) = 0$ and $v(f_i(x)) + (i-1)\delta > 0$ for all $i \geq 2$.

Indeed, $v(f(c)-c) = v(f(x)-x) > \delta$ for c near x, so $v(f(x)-f(c)) = v(x-c)$, and (1) holds for infinitely many values $v(x-c)$, by Observation 1.1. All this applies equally to every $t \in H$ with $M(t)^{\mathrm{h}} = H$ in place of x.

(i): Put $N := M(y)^{\mathrm{h}}$. As $y \notin M$, the extension H/N is finite. We show first that N is dense in H. Let $r \in \mathbb{R}$, and take $f \in M[X]$ with $v(f(x)-y) > \max\{r,\delta\}$. Then $v(f(x)-x) > \delta$, so (3) applies. Take $c \in M$ near x such that $\gamma := v(x-c)$ satisfies $v(f(x)-f(c)) = \gamma$, and $v(f_i(c)) = v(f_i(x))$ for all i, and $v(f_i(x)) + (i-1)\gamma > 0$ for all $i \geq 2$. Take $s \in M$ with $v(s) = \gamma$. Then

$$P(T) := \frac{f(c+sT)-y}{f_1(c)s} = a_0 + T + \sum_{i\geq 2} a_i T^i, \qquad a_0 = \frac{f(c)-y}{f_1(c)s}, \quad a_i = \frac{f_i(c)s^{i-1}}{f_1(c)},$$

where $v(a_0) = 0$, as $v(f(c)-y) = v(f(c)-f(x)) = \gamma$, and $v(a_i) > 0$ for $i \geq 2$. By Hensel's Lemma [EP05, Theorem 4.1.3(4)], applied at $-a_0$, P has a zero $T_0 \in \mathcal{O}_N$. For $T_x := (x-c)/s \in \mathcal{O}_H$ we have $P(T_x) - P(T_0) = (T_x - T_0)(1+m)$ with $v(m) > 0$. So $z := c + sT_0 \in N$ satisfies $v(x-z) = \gamma + v(P(T_x)) = v(f(x)-y) > r$. Thus x, hence $M(x)$, hence H, lies in the closure of N. So the separable closure of N in H lies in the completion of the henselian field N, hence equals N [War89, Theorem 32.19], and H/N is purely inseparable. Suppose that $[H : N] = p^k$ with $k \geq 1$. Then $u := x^{p^k} \in N$, and $N = M(u)^{\mathrm{h}}$, since $H = M(u)^{\mathrm{h}}(x)$ has degree at most p^k over $M(u)^{\mathrm{h}} \subseteq N$. So there is $g \in M[X]$ with $v(g(u)-y) > \delta$, and then $G(X) := g(X^{p^k})$ satisfies $v(G(x)-x) > \delta$. By (3), $v(G_1(x)) = 0$, which is absurd, as $G_1 = G' = 0$. So $H = N$.

(ii): Put $\delta_w := \sup v(w-M) < \infty$. Take $g \in M[X]$ with $v(g(w)-x) > \delta$, and then $f \in M[X]$ with $v(f(x)-w) > \delta_w$, so close that $v(g(f(x))-g(w)) > \delta$. For $b \in M$ we have $v(b-w) = v(b-f(x))$ and $v(g(b)-x) = v(g(b)-g(w))$, by Observation 1.1. So $\delta_w = \sup v(f(x)-M) = \beta_f + h_f\delta$, by (2). So by (1) applied to f at x, and then to g at w (as $v(f(c)-w)$ tends to δ_w),

$$v(x-g(f(c))) = \beta_g + h_g\big(\beta_f + h_f v(x-c)\big) \quad \text{for } c \text{ near } x.$$

On the other hand $v(g(f(x))-x) > \delta$, so $v(g(f(c))-c) > \delta$ for c near x, whence $v(x-g(f(c))) = v(x-c)$. By Observation 1.1, $h_g h_f = 1$. So $h_f = 1$, i.e. $v(f_i(x)) + i\delta > v(f_1(x)) + \delta$ for $i \geq 2$. Take $c \in M$ near x with $v(f_i(c)) = v(f_i(x))$ for all i, and $v(f_i(x)) + i\,v(x-c) > v(f_1(x)) + \delta$ for $i \geq 2$. Then $e := f_1(c)$ and $c' := f(c) - ec$ satisfy $v(f(x)-ex-c') = v\big(\sum_{i\geq 2} f_i(c)(x-c)^i\big) > v(e)+\delta$, while $v(w-f(x)) > \delta_w = v(e)+\delta$. So e and c' are as required.

(iii): As v has rank 1, E is dense in $E^{\mathrm{h}} = H$, so some $y \in E$ satisfies $v(y-x) > \delta$, and **(i)** applies. □

Proposition 3.16 (Main Proposition in case (IMM), cf. [Kuh19a, Proposition 5.2]). *Let (K,v) be a separably closed valued field of residue characteristic $p > 0$ and rank 1, and let $(F/K,v)$ be an immediate separable henselized function field of transcendence degree 1, say $F = E^{\mathrm{h}}$. Then $F = K(y)^{\mathrm{h}}$ for some $y \in E$. That is, $(E/K,v)$ is henselian rational.*

Proof. By Fact 3.13, (K,v) is henselian, Γ_K is divisible and k_K is algebraically closed. Moreover, (K,v) is separably algebraically maximal and separably tame, hence dense in K^{perf} (Corollary 1.13), and in mixed characteristic it is algebraically closed. Thus (K,v) satisfies ($\mathbf{H_{IMM}}$).

Since E/K is separable, it is separably generated [ML39]. Choose a separating element $x \in E$, so that $E/K(x)$ and hence $F/K(x)^{\mathrm{h}}$ are finite separable. If $F = K(x)^{\mathrm{h}}$ we are done, so suppose not.

Claim 3.16.1 ([Kuh19a, Lemma 5.1])**.** For every $y \in F$ transcendental over K, each nontrivial finite separable extension of $K(y)^{\mathrm{h}}$ is a tower of Galois extensions of degree p.

Proof of claim. Since $(K(y)/K,v)$ is immediate, the value group $\Gamma_{K(y)} = \Gamma_K$ is divisible and the residue field $k_{K(y)} = k_K$ is algebraically closed, so the separable closure of $K(y)^{\mathrm{h}}$ is an immediate extension of it. By the Lemma of Ostrowski, every finite subextension has p-power degree, so the separable closure is a p-extension. By the theory of p-groups and the Galois correspondence, every finite subextension is a tower of Galois extensions of degree p. ■claim

By the Claim, write $K(x)^{\mathrm{h}} = F_0 \subset F_1 \subset \cdots \subset F_r = F$ with each F_{j+1}/F_j Galois of degree p. We prove by induction that $F_j = K(y_j)^{\mathrm{h}}$ for some $y_j \in F_j$, starting from $y_0 = x$. Given this for j: the element y_j is transcendental over K and $(K(y_j)/K,v)$ is immediate (as a subextension of $(F/K,v)$). Since K is separably closed, the split case of Proposition 3.12 cannot occur (K has no Artin–Schreier extensions). Hence Propositions 3.10 and 3.12, applied to $F_{j+1}/F_j = F_{j+1}/K(y_j)^{\mathrm{h}}$, produce $y_{j+1} \in F_{j+1}$ with $F_{j+1} = K(y_{j+1})^{\mathrm{h}}$. Taking $y := y_r \in F$ we obtain $F = K(y)^{\mathrm{h}}$.

Finally, y may be taken in E: if F embeds into (K^c,v) over K, then $F = K(x)^{\mathrm{h}}$ for the separating element $x \in E$ by Proposition 3.14. Otherwise $\sup v(y-K) < \infty$, as otherwise $K(y)$, and hence $F = K(y)^{\mathrm{h}}$, would embed into the henselian field (K^c,v). So Lemma 3.15**(iii)** applies, with $M := K$. □

3.4. **The reduction.** We now reduce Theorem 2 to Proposition 3.16, following [Kuh19a, §5]: descent along tame extensions, then finite rank, then arbitrary rank, then higher transcendence degree. Throughout, v is extended to $\widetilde{F}$ and all henselizations and composita are taken inside $(\widetilde{F},v)$.

Lemma 3.17 (Composita)**.** *Let $(F/K,v)$ be a henselized function field, $F = E^{\mathrm{h}}$ with E finitely generated over K, and let $(L/K,v)$ be algebraic.*

(i) *$FL = (EL)^{\mathrm{h}}$. In particular $(FL/L,v)$ is a henselized function field, of the same transcendence degree.*
(ii) *If E/K and L/K are separable, then so is EL/L, and hence FL/L.*
(iii) *If $(F/K,v)$ is immediate, then $(K^{\mathrm{sep}}F/K^{\mathrm{sep}},v)$ is immediate.*

Proof. **(i)**: FL is algebraic over the henselian field F, hence henselian, and contains EL. Conversely, $(EL)^{\mathrm{h}}$ is henselian and contains E and L, hence contains $E^{\mathrm{h}} = F$ and so FL. **(ii)**: a separating transcendence basis T of E/K remains algebraically independent over L, and EL is generated over $L(T)$ by elements separably algebraic over $K(T)$. **(iii)**: $v(K^{\mathrm{sep}}F)$ lies in the divisible hull of $vF = \Gamma_K$, which equals $\Gamma_{K^{\mathrm{sep}}}$ by Fact 3.13, and $k_{K^{\mathrm{sep}}F}$ is algebraic over $k_F = k_K$, hence contained in $\widetilde{k_K} = k_{K^{\mathrm{sep}}}$. □

Proposition 3.18 (Tame descent, cf. [KV14, Theorem 14.5] and [Kuh19a, §5.2])**.** *Let (K,v) be separably tame of rank 1, and let $(F/K,v)$ be an immediate separable henselized function field of transcendence degree 1, say $F = E^{\mathrm{h}}$, which admits no valuation-preserving K-embedding into (K^c,v). If $FL = L(y)^{\mathrm{h}}$ for some separably algebraic extension L/K and some $y \in FL$, then $F = K(y')^{\mathrm{h}}$ for some $y' \in E$.*

This is [KV14, Theorem 14.5], stated there for algebraically maximal (K,v). As observed in [Kuh19a, §5.2], the proof uses only that approximation types of elements transcendental over K are transcendental, which holds whenever (K,v) is separably algebraically maximal, by Lemma 1.14**(ii)**. It is the precise analogue for henselian rationality of pulling defectlessness down through tame extensions in Section 2.

Proof. We have $y \in FL_1$ for a finite subextension L_1 of L/K, and the finitely many generators of E over K lie in $L_2(y)^{\mathrm{h}}$ for a finite subextension $L_2 \supseteq L_1$. Replacing L by the normal hull of L_2 over K, we may assume that L/K is finite and Galois, by Lemma 3.17**(i)**. As K is separably tame, the finite subextensions of K^{sep}/K are tame over K, hence over L. So L is separably tame, in particular separably algebraically maximal, and L/K is tame.

Choose $\omega_1 = 1, \omega_2, \dots, \omega_f \in \mathcal{O}_L$ whose residues form a basis of k_L/k_K, and $\pi_1 = 1, \pi_2, \dots, \pi_e \in L$ whose values represent Γ_L/Γ_K, so that $ef = [L : K]$. For every immediate extension $(F'/K,v)$ inside $\widetilde{F}$ we have $\Gamma_{F'} = \Gamma_K$ and $k_{F'} = k_K$, so by [EP05, Lemma 3.2.2],

$$(*) \qquad v\Big(\sum_{i,j} a_{ij}\omega_i\pi_j\Big) = \min_{i,j} v(a_{ij}\omega_i\pi_j) \quad \text{for all } a_{ij} \in F'.$$

In particular $[F'L : F'] = [L : K]$, and $(F'L/L, v)$ is immediate.

First, $\delta := \sup v(y - L) < \infty$. Otherwise L is dense in $L(y)$, hence in $L(y)^{\mathrm{h}} = FL$. But let $a \in F$ and $b = \sum_{i,j} c_{ij}\omega_i\pi_j \in L$, with $c_{ij} \in K$. By $(*)$ for $F' = F$ and Observation 1.1**(ii)** for FL/F, we have $v(b-a) \leq v(b-c_{11})$, hence $v(a - c_{11}) \geq v(a - b)$. So K would be dense in F, and F would embed into K^c. Let $G := \mathrm{Gal}(FL/F)$. As $[FL : F] = [L : K]$, restriction $G \to \mathrm{Gal}(L/K)$ is bijective. Each $\rho \in G$ preserves v, as F is henselian, and $\rho(L) = L$, so ρ maps $L(y)^{\mathrm{h}}$ onto $L(\rho(y))^{\mathrm{h}} = FL$. By Lemma 3.15**(ii)**, applied over L, there are $e_\rho \in L^\times$ and $c_\rho \in L$ with $v(\rho(y) - e_\rho y - c_\rho) > v(e_\rho) + \delta$. As $\sup v(\rho(y) - L) = \delta$, it follows that $v(e_\rho) = 0$. We claim that there is $d \in L^\times$ with $v\big(\sum_{\rho\in G} \rho(d)e_\rho\big) = v(d)$. Then $y' := \mathrm{Tr}_{FL/F}(dy) = \sum_\rho \rho(d)\rho(y) \in F$ satisfies $v(y' - ey - c) > v(d) + \delta = v(e) + \delta$, for $e := \sum_\rho \rho(d)e_\rho$ and $c := \sum_\rho \rho(d)c_\rho$. As $L(ey + c)^{\mathrm{h}} = FL$ and $\sup v(ey + c - L) = v(e) + \delta$, Lemma 3.15**(i)**, applied over L, gives $L(y')^{\mathrm{h}} = FL$.

For the claim, let $\chi_\rho(d)$ be the residue of $\rho(d)/d$, for $\rho \in G$ and $d \in L^\times$. Each $\chi_\rho : L^\times \to k_L^\times$ is a homomorphism, and $\chi_{\rho\tau}(d) = \chi_\rho(\tau(d))\chi_\tau(d)$. By Artin's theorem on the independence of characters [Lan87, VI, Theorem 4.1], it suffices that the χ_ρ are pairwise distinct: then $\sum_\rho \overline{e_\rho}\chi_\rho(d) \neq 0$ for some d. By the cocycle identity, $\chi_\rho = \chi_\tau$ if and only if $\chi_{\rho\tau^{-1}} = 1$, and the $\sigma \in G$ with $\chi_\sigma = 1$ form a subgroup. So suppose that $\chi_\sigma = 1$ for some $\sigma \in G$ of prime order q, and let L_0 be the fixed field of $\sigma|_L$. Then $T := \sigma - 1$ satisfies $v(T(d)) > v(d)$ for all $d \in L^\times$, and $\sum_{j=1}^{q} \binom{q}{j} T^j = \sigma^q - 1 = 0$ on L. If $q \neq p$, then $v(qT(d)) = v(T(d)) < v(T^j(d))$ for $d \in L \setminus L_0$ and $j \geq 2$, a contradiction. If $q = p$, then $f(L/L_0) = p$, as L/L_0 is tame. But for $d \in \mathcal{O}_L$ the polynomial $\prod_{j<p}(X - \sigma^j(d)) \in \mathcal{O}_{L_0}[X]$ has residue $(X - \overline{d})^p$, so $\overline{d}$ is purely inseparable over k_{L_0}, hence lies in k_{L_0}, as k_L/k_{L_0} is separable, a contradiction.

Now comes the descent step. We have $K(y')^{\mathrm{h}} \subseteq F$, and $LK(y')^{\mathrm{h}} = L(y')^{\mathrm{h}} = FL$, as in Lemma 3.17**(i)**. By the degree count for $F' = K(y')^{\mathrm{h}}$ and for $F' = F$, we get $[FL : K(y')^{\mathrm{h}}] = [L : K] = [FL : F]$, so $F = K(y')^{\mathrm{h}}$. Finally $\sup v(y' - K) < \infty$, as otherwise $K(y')$, and hence F, would embed into K^c. So Lemma 3.15**(iii)**, applied over K, gives a generator in E. This is the analogue for henselian rationality of pulling defectlessness down through tame extensions in Section 2. □

Proposition 3.19 (Rank 1, separably tame base, cf. [Kuh19a, Proposition 5.3])**.** *Let (K, v) be separably tame of rank 1 and let $(F/K, v)$ be an immediate separable henselized function field of transcendence degree 1, say $F = E^{\mathrm{h}}$. Then $F = K(y)^{\mathrm{h}}$ for some $y \in E$.*

Proof. If $\mathrm{char} k_K = 0$, then $F = K(x)^{\mathrm{h}}$ for every separating element x of E/K, which exists by [ML39], as $F/K(x)^{\mathrm{h}}$ is finite, immediate and defectless by Ostrowski's lemma. So let $\mathrm{char} k_K = p > 0$. If F embeds into (K^c, v) over K, apply Proposition 3.14, a separating element existing by [ML39]. Otherwise, set $L := K^{\mathrm{sep}}$. By Lemma 3.17, $(FL/L, v)$ is an immediate separable henselized function field of transcendence degree 1 and rank 1 over the separably closed field (L, v), so Proposition 3.16 gives $FL = L(y)^{\mathrm{h}}$. Proposition 3.18 concludes. In both cases the generator lies in E. □

Lemma 3.20 (Separating elements with prescribed value and residue, [Kuh19a, Lemma 5.4])**.** *Let F/K be a separable function field of transcendence degree 1 and w a nontrivial valuation on F. For every $x \in F^\times$ there is a separating element y of F/K with $w(y) = w(x)$, and with $\overline{y}^w = \overline{x}^w$ in case $w(x) = 0$.*

Proof. Choose a separating element z of F/K and $a \in F^\times$ with $w(a) > w(x)$ and $w(az) > w(x)$. Then $K(x, z, a) = K(x,\ x + az,\ x + a)$, and each of x, $x + az$, $x + a$ has the same value as x, and the same residue when $w(x) = 0$. The extension $K(x, z, a)/K$ is separably generated (it lies between $K(z)$ and F), so by [Lan87, VIII, §4, Proposition 4.8] one of the three generators, call it y, is a separating element of $K(x, z, a)/K$. Since $F/K(z)$ is separably algebraic, y is then a separating element of F/K. □

Lemma 3.21 (Composite places, [Kuh19a, Lemma 5.5])**.** *Let (L, P) be a valued field with $P = Q_1Q_2$ the composition of a place Q_1 on L and a place Q_2 on LQ_1.*

- **(i)** *Let L'/L be an extension with compatible decomposition $P = Q_1Q_2$ on L'. Then $(L'/L, P)$ is immediate if and only if $v_{Q_1}L' = v_{Q_1}L$ and $(L'Q_1/LQ_1, Q_2)$ is immediate.*
- **(ii)** *(L, P) is henselian if and only if (L, Q_1) and (LQ_1, Q_2) are henselian.*
- **(iii)** *The extension $(L^{\mathrm{h}(P)}/L^{\mathrm{h}(Q_1)}, Q_1)$ is tame, and $(L^{\mathrm{h}(P)})Q_1 = (LQ_1)^{\mathrm{h}(Q_2)}$.*

Proof. **(i)** follows from the exact sequence $0 \to v_{Q_2}(LQ_1) \to v_PL \to v_{Q_1}L \to 0$ of the convex subgroup associated with Q_1, together with $L'P = (L'Q_1)Q_2$. **(ii)** is [War89, Theorem 32.15]. **(iii)**: by [EP05, Theorem 5.2.7], the residue map induces a correspondence between subextensions of the absolute inertia field $L^{i(Q_1)}$ of $(L^{\mathrm{h}(Q_1)}, Q_1)$ and separably algebraic extensions of LQ_1. Let (E, Q_1) correspond to $(LQ_1)^{\mathrm{h}(Q_2)}$. As a subextension of the inertia field, $(E/L^{\mathrm{h}(Q_1)}, Q_1)$ is tame. By **(ii)**, (E, P) is henselian, so $E \supseteq L^{\mathrm{h}(P)}$. Conversely,

$(L^{\mathrm{h}(P)}, Q_1)$ is henselian and $(L^{\mathrm{h}(P)}Q_1, Q_2)$ is henselian by **(ii)**, so $L^{\mathrm{h}(P)}Q_1 \supseteq (LQ_1)^{\mathrm{h}(Q_2)} = EQ_1$, and the correspondence yields $E \subseteq L^{\mathrm{h}(P)}$. □

Proposition 3.22 (Finite rank, cf. [Kuh19a, Proposition 5.6])**.** *Let (K, v) be separably tame of finite rank and let $(F/K, v)$ be an immediate separable henselized function field of transcendence degree 1, say $F = E^{\mathrm{h}}$. Then $F = K(x)^{\mathrm{h}}$ for some $x \in E$.*

Proof. Let P be the place of v. Since the rank is finite, there is a decomposition $P = P_1P_2P_3$ with P_1, P_3 possibly trivial and P_2 of rank 1, such that $\mathrm{trdeg}(FP_1/KP_1) = 1$ and $\mathrm{trdeg}(FP_1P_2/KP_1P_2) = 0$. By [Kuh16, Lemma 3.14], (K, P_1) and (KP_1, P_2) are separably tame, (KP_1, P_2) is tame if P_1 is nontrivial, and (KP_1P_2, P_3) is tame. By Lemma 3.21**(i)** applied twice, immediacy of $(F/K, P)$ gives $v_{P_1}F = v_{P_1}K$, $v_{P_2}(FP_1) = v_{P_2}(KP_1)$, and that $(FP_1P_2/KP_1P_2, P_3)$ is immediate algebraic. The tame field (KP_1P_2, P_3) is algebraically maximal, so $FP_1P_2 = KP_1P_2$, and hence $(FP_1/KP_1, P_2)$ is immediate, of transcendence degree 1 and rank 1, and finitely generated by [Kuh16, Corollary 2.3]. It is separable, since KP_1 is perfect when P_1 is nontrivial ((KP_1, P_2) is tame), and equals E/K when P_1 is trivial. By Proposition 3.19, applied to $((FP_1)^{\mathrm{h}(P_2)}/KP_1, P_2)$ with model FP_1,

$$(FP_1)^{\mathrm{h}(P_2)} = KP_1(\xi)^{\mathrm{h}(P_2)}$$

for some $\xi \in FP_1$ transcendental over KP_1. Lift ξ to F and, by Lemma 3.20 applied with $w = v_{P_1}$ (or with $w = v$ if P_1 is trivial), choose a separating element $x \in E$ with $v_{P_1}(x) = 0$ and $xP_1 = \xi$.

Since $(FP_1P_2, P_3) = (KP_1P_2, P_3)$ is henselian, Lemma 3.21**(ii)** gives $F^{\mathrm{h}(P)} = F^{\mathrm{h}(P_1P_2)}$ and $K(x)^{\mathrm{h}(P)} = K(x)^{\mathrm{h}(P_1P_2)}$. Then Lemma 3.21**(iii)** yields

$$K(x)^{\mathrm{h}(P)}P_1 \subseteq F^{\mathrm{h}(P)}P_1 = (FP_1)^{\mathrm{h}(P_2)} = KP_1(xP_1)^{\mathrm{h}(P_2)} \subseteq (K(x)P_1)^{\mathrm{h}(P_2)} = K(x)^{\mathrm{h}(P)}P_1,$$

so equality holds throughout. If P_1 is trivial, this reads $F = F^{\mathrm{h}(P)} = K(x)^{\mathrm{h}(P)}$ and we are done. If P_1 is nontrivial, the displayed equality of residue fields, together with $v_{P_1}K \subseteq v_{P_1}K(x)^{\mathrm{h}(P)} \subseteq v_{P_1}F^{\mathrm{h}(P)} = v_{P_1}K$ (Lemma 3.21**(i)**), shows that $(F^{\mathrm{h}(P)}/K(x)^{\mathrm{h}(P)}, P_1)$ is immediate. It is finite and separable since x is separating. Now (K, P_1) is separably tame, hence separably defectless, and xP_1 is transcendental over KP_1, so $(K(x)/K, P_1)$ is valuation regular without transcendence defect and Corollary 2.32 (with $\Gamma_K = \Gamma_{K(x)}$ for P_1, so the cofinality hypothesis holds) shows that $(K(x), P_1)$ is separably defectless. By [End72, Theorem (18.2)] so is $(K(x)^{\mathrm{h}(P_1)}, P_1)$, and since $(K(x)^{\mathrm{h}(P)}/K(x)^{\mathrm{h}(P_1)}, P_1)$ is tame (Lemma 3.21**(iii)**), also $(K(x)^{\mathrm{h}(P)}, P_1)$ is separably defectless [Kuh16, Proposition 2.12]. A finite separable immediate extension of a henselian separably defectless field is trivial, whence $F = F^{\mathrm{h}(P)} = K(x)^{\mathrm{h}(P)}$. □

Proposition 3.23 (Arbitrary rank, cf. [Kuh19a, Proposition 5.7])**.** *Let (K, v) be separably tame and let $(F/K, v)$ be an immediate separable henselized function field of transcendence degree 1, say $F = E^{\mathrm{h}}$. Then $F = K(x)^{\mathrm{h}}$ for some $x \in E$.*

Proof. By [Kuh16, Corollaries 3.8 and 3.16] there are a separably tame subfield $K_0 \subseteq K$ of finite rank and a function field E_0/K_0 of transcendence degree 1, which may be taken separable, with $E = E_0K$, $K_0v = Kv$, vK/vK_0 torsion free, and vK_0 cofinal in vE_0. In particular (E_0, v) has finite rank, since vK_0 has finite rank and is cofinal in vE_0. If $(E_0/K_0, v)$ is immediate, then Proposition 3.22 applied to $(E_0^{\mathrm{h}}/K_0, v)$ gives $E_0^{\mathrm{h}} = K_0(x)^{\mathrm{h}}$ with $x \in E_0 \subseteq E$, and by Lemma 3.17**(i)**,

$$F = (E_0K)^{\mathrm{h}} = (E_0^{\mathrm{h}}K)^{\mathrm{h}} = (K_0(x)^{\mathrm{h}}K)^{\mathrm{h}} = (K_0(x)K)^{\mathrm{h}} = K(x)^{\mathrm{h}}.$$

Otherwise, $E_0v = K_0v$ (sandwiched between K_0v and $Kv = K_0v$), so $vE_0 \neq vK_0$. The group vE_0/vK_0 is torsion free (as $vE_0 \subseteq vK$ and vK/vK_0 is torsion free) and finitely generated of rational rank 1 [Kuh16, Corollary 2.3], so $vE_0 = vK_0 \oplus \mathbb{Z}vx$ for a suitable $x \in E_0$, which by Lemma 3.20 may be taken separating. Then $(E_0/K_0(x), v)$ is finite, separable and immediate, $(K_0(x)/K_0, v)$ is valuation regular without transcendence defect, and Corollary 2.32 (applicable since vK_0 is cofinal in $vE_0 \supseteq vK_0(x)$) together with [End72, Theorem (18.2)] makes $(K_0(x)^{\mathrm{h}}, v)$ separably defectless. Hence $E_0^{\mathrm{h}} = K_0(x)^{\mathrm{h}}$, and the displayed chain concludes as before. □

3.5. **Proof of Theorem 2.** A tame field is perfect, so F/K is separable, and (K, v) is in particular separably tame. Writing $F = E^{\mathrm{h}}$, Proposition 3.23 yields $x \in E$ with $F = K(x)^{\mathrm{h}}$, i.e. $(E/K, v)$ is henselian rational in the sense of Definition 3.2.

3.6. **Higher transcendence degree.** Theorem 2 concerns transcendence degree 1. We record the general case, in which one passes to a finite immediate extension.

Proposition 3.24 (Descending to a finite extension, cf. [Kuh19a, Proposition 5.8])**.** *Let $(F/K, v)$ be a henselized function field, $F = E^{\mathrm{h}}$, and let K'/K be an algebraic extension such that $FK' = K'(T)^{\mathrm{h}}$ for a finite transcendence*

basis $T \subseteq FK'$. Then there is a finite subextension L/K of K'/K with $T \subseteq FL$ and $FL = L(T)^{\mathrm{h}}$, and the same holds with L replaced by any algebraic extension of L.

Proof. By Lemma 3.17**(i)** (applied over $K(T)$), $K'(T)^{\mathrm{h}} = K(T)^{\mathrm{h}}K' = \bigcup_L L(T)^{\mathrm{h}}$, a directed union over the finite subextensions L/K of K'/K. Let S be a finite generating set of E/K. Choose L finite with $T \subseteq FL$ and $S \subseteq L(T)^{\mathrm{h}}$. Then $EL = L(S) \subseteq L(T)^{\mathrm{h}}$, so $FL = (EL)^{\mathrm{h}} \subseteq L(T)^{\mathrm{h}}$. Conversely, $FL \supseteq L(T)$ is henselian, so $FL \supseteq L(T)^{\mathrm{h}}$, giving equality. Both memberships $T \subseteq FL$ and $S \subseteq L(T)^{\mathrm{h}}$ persist when L is enlarged. □

Proposition 3.25 (Existence of ambient separably tame fields)**.** *Let (K, v) be separably tame of residue characteristic $p > 0$ with v nontrivial, and let $(F/K, v)$ be an immediate separable extension. Then (F, v) admits an immediate separably algebraic extension (N, v) which is a separably tame field.*

Proof. By Lemma 1.8, $vF = vK$ is p-divisible and $Fv = Kv$ is perfect. Let (N, v) be a maximal immediate separably algebraic extension of (F, v), which exists by Zorn's Lemma. Then (N, v) is separably algebraically maximal with vN p-divisible and Nv perfect, hence separably tame, again by Lemma 1.8. □

Proposition 3.26 (Higher transcendence degree, cf. [Kuh19a, Theorem 1.3])**.** *Let (K, v) be separably tame and let $(F/K, v)$ be an immediate separable henselized function field of transcendence degree $n \geq 1$. Let (N, v) be a separably tame field which is an immediate separable extension of (F, v). Such (N, v) exist by Proposition 3.25. Then there are a finite immediate separable extension $(F_1/F, v)$ with $F_1 \subseteq N$ and elements $x_1, \dots, x_n \in F_1$ such that $F_1 = K(x_1, \dots, x_n)^{\mathrm{h}}$.*

Proof. Induction on n. For $n = 1$ take $F_1 = F$ and apply Proposition 3.23. Let $n \geq 2$ and write $F = E^{\mathrm{h}}$. Choose a separating transcendence basis $\{t_1, \dots, t_{n-1}, t\}$ of E/K [ML39], and let N' be the relative algebraic closure of $K(t_1, \dots, t_{n-1})$ in N. As a subextension of the separable $N/K(t_1, \dots, t_{n-1})$, the extension $N'/K(t_1, \dots, t_{n-1})$ is separable, and by [Kuh19a, Lemma 5.9] (refining [Kuh16, Lemma 3.15]), (N', v) is a separably tame field. This is also Lemma 1.10, as N/N' is separable by **(P4)** and $(N/K, v)$ is immediate. The extension $(FN'/N', v)$ is an immediate separable henselized function field of transcendence degree 1, by Lemma 3.17. Indeed, t remains transcendental over N', and immediacy follows since everything lies in the immediate extension $(N/K, v)$. So Proposition 3.23 gives $FN' = N'(x)^{\mathrm{h}}$ with $x \in EN'$. By Proposition 3.24, applied over the base $K(t_1, \dots, t_{n-1})$ with $K' = N'$, there is a finite extension L of $K(t_1, \dots, t_{n-1})$ inside N' such that $FL' = L'(x)^{\mathrm{h}}$ for $L' = L$ and every algebraic extension L' of L. Now $(L/K, v)$ is an immediate separable function field of transcendence degree $n - 1$, and $(N'/L^{\mathrm{h}}, v)$ is an immediate *separable* extension: indeed N'/L is separably algebraic, because every element of N' is separably algebraic over $K(t_1, \dots, t_{n-1})$ and minimal polynomials over L divide those over $K(t_1, \dots, t_{n-1})$. By the induction hypothesis applied to $(L^{\mathrm{h}}/K, v)$ inside (N', v), there is a finite immediate separable extension $(L_1/L^{\mathrm{h}}, v)$ with $L_1 \subseteq N'$ and $L_1 = K(x_1, \dots, x_{n-1})^{\mathrm{h}}$. By the stability clause of Proposition 3.24, $FL_1 = L_1(x)^{\mathrm{h}}$. Setting $F_1 := FL_1 \subseteq N$ we obtain

$$F_1 = L_1(x)^{\mathrm{h}} = \big(K(x_1, \dots, x_{n-1})^{\mathrm{h}}(x)\big)^{\mathrm{h}} = K(x_1, \dots, x_{n-1}, x)^{\mathrm{h}}.$$

Finally, F_1/F is finite (L_1 is finite over L^{h} and L is finite over $K(t_1, \dots, t_{n-1}) \subseteq F$), immediate ($F_1 \subseteq N$), and separable ($L_1/K(t_1, \dots, t_{n-1})$ is separably algebraic). □

Remark 3.27. In the notation of the proof, $F_1 = FL_1$, where $L_1 \subseteq N$ is a finite extension of $K(t_1, \dots, t_{n-1})^{\mathrm{h}}$. Thus $(F/K, v)$ becomes henselian rational after a finite extension of the constants of the subfield $K(t_1, \dots, t_{n-1})$. We do not know whether one may take $F_1 = F$ when $n \geq 2$. The argument above does not give this: the case of transcendence degree 1 (Proposition 3.23) requires a separably tame base, so the generator x is found only over N', and descending from N' yields a finite extension L of $K(t_1, \dots, t_{n-1})$ inside N', which need not lie in F. Kuhlmann states the general case in the same form [Kuh19a, Theorem 1.3]. One may take $F_1 = F$ when $n = 1$ (Theorem 2), when (K, v) has residue characteristic 0 or is finitely ramified [Kuh19a, Theorem 2.2], and when F/K is a separable subextension of K^c/K, where K^c is the completion [Kuh19a, Theorem 2.3].

4. Separably tame valued fields: the embedding theorem and Ax–Kochen/Ershov principles

We use **(P1)**–**(P5)**, **(K1)**–**(K3)** and the notions of Section 1. Let **STVF** be the $\mathfrak{L}_{\mathrm{val}}$-theory of separably tame valued fields, including the trivially valued ones, and let $\mathbf{STVF}^{(\mathrm{EQ})}$ be its extension by axioms saying that the field and its residue field have the same characteristic. The class of separably tame valued fields is elementary. Indeed, in residue characteristic 0, separably tame means henselian, by Ostrowski's lemma. In residue characteristic $p > 0$, by Lemma 1.8, it means that either the valuation is trivial, or the field is henselian and separably algebraically maximal, with p-divisible value group and perfect residue field. Henselianity is expressed by an axiom scheme [EP05, Theorem 4.1.3]. For a henselian (K, v) and a separable irreducible $g \in$

$K[X]$, the valued field $K[X]/(g)$ is interpretable in (K,v), uniformly in the coefficients of g: its valuation ring is the integral closure of $\mathcal{O}_K$ [EP05, Corollary 3.1.4], i.e. the set of elements whose characteristic polynomial over K has coefficients in $\mathcal{O}_K$. So separable algebraic maximality is expressed by saying, for each $n \geq 2$, that no such extension of degree n is immediate.

(S) We use saturation in the following standard form, cf. [Kuh16, §5.1]. Let $(L/M,v)$ be an extension of valued fields, and let ι be an embedding of (M,v) into an $|L|^+$-saturated valued field (K^*,v^*). Let Σ be a set of formulas in variables x_a, $a \in L$, with at most $|L|$ parameters from K^*, containing the atomic diagram of the valued field L over M, transported along ι. If for each finitely generated subextension F of L/M, ι extends to F satisfying the formulas of Σ in the variables x_a, $a \in F$, then Σ is finitely satisfiable, hence realized, so ι extends to L satisfying Σ. For example, an embedding respects given embeddings $\rho : \Gamma_L \to \Gamma_{K^*}$ and $\sigma : k_L \to k_{K^*}$ if and only if it satisfies the formulas $v(x_a c_a^{-1}) = 0$ and $v(x_a - d_a) > 0$, where $v^*(c_a) = \rho(v(a))$ and $\overline{d_a} = \sigma(\overline{a})$.

4.1. **Extensions without transcendence defect.**

Lemma 4.1 (Embedding strongly inertially generated extensions, [Kuh16, Lemma 5.6])**.** *Let $(F/K,v)$ be a strongly inertially generated henselized function field, and let (K^*,v^*) be a henselian extension of (K,v). Let $\rho : \Gamma_F \to \Gamma_{K^*}$ and $\sigma : k_F \to k_{K^*}$ be embeddings over Γ_K and k_K, respectively. Then there is an embedding $\iota : (F,v) \to (K^*,v^*)$ over K that respects ρ and σ, i.e. $v^*(\iota(a)) = \rho(v(a))$ and $\overline{\iota(a)} = \sigma(\overline{a})$ for all $a \in F$, the latter whenever $v(a) \geq 0$.*

Proof. Let $t = (t_1,\dots,t_r)$ and $u = (u_1,\dots,u_s)$ be as in the definition, so that $F/K(t,u)^{\mathrm{h}}$ is inert. Choose $t_i' \in K^*$ with $v^*(t_i') = \rho(v(t_i))$, and units $u_j' \in K^*$ with $\overline{u_j'} = \sigma(\overline{u_j})$. As ρ and σ are embeddings over Γ_K and k_K, Lemma 1.2 applies to t',u' as well as to t,u. Its formula shows that $t \mapsto t'$, $u \mapsto u'$ defines a K-isomorphism $\iota : K(t,u) \to K(t',u')$ with $v^*(\iota(a)) = \rho(v(a))$. So ι is an isomorphism of valued fields. The induced map of residue fields agrees with σ on k_K and on the $\overline{u_j}$, which generate $k_{K(t,u)}$ over k_K, so it is σ. By the universal property of the henselization, ι extends to $K(t,u)^{\mathrm{h}}$ [EP05, Theorem 5.2.2], still respecting ρ and σ, as $K(t,u)^{\mathrm{h}}/K(t,u)$ is immediate [EP05, Theorem 5.2.5]. Choose $a \in \mathcal{O}_F$ with $k_F = k_{K(t,u)}(\overline{a})$. Then $F = K(t,u)^{\mathrm{h}}(a)$ by the Fundamental Inequality, as $F/K(t,u)^{\mathrm{h}}$ is inert. The minimal polynomial g of a is integral, and its reduction is the minimal polynomial of $\overline{a}$, by comparing degrees, so it is separable. By Hensel's Lemma, $\iota(g)$ has a root $a' \in K^*$ with $\overline{a'} = \sigma(\overline{a})$, and $a \mapsto a'$ extends ι to F. As $K(t,u)^{\mathrm{h}}$ is henselian, $v^* \circ \iota$ is equivalent to v on F. So ι respects ρ, as $\Gamma_F = \Gamma_{K(t,u)}$, and σ, as $k_F = k_{K(t,u)}(\overline{a})$. □

Proposition 2.31 needs a defectless base field. A separably tame field (K,v) with v nontrivial and K imperfect is not defectless: it is dense in $K^{1/p}$ by Corollary 1.13, so $K(a^{1/p})/K$ is immediate of degree p for $a \in K \setminus K^p$. So we follow [KP16, §3.2]: we apply Proposition 2.31 over the perfect hull K^{perf}, and then pull back. Kuhlmann and Pal pull back from the completion, with the Implicit Function Theorem [PZ78, Theorem 7.4]. We pull back from the perfect hull, with Newton's Lemma.

Lemma 4.2 (Pulling back from the perfect hull, cf. [KP16, Lemma 3.6])**.** *Let (K^*,v^*) be a henselian valued field of characteristic $p > 0$, let $H := (K^*)^{\mathrm{perf}}$, and suppose that K^* is dense in (H,v^*). Let $K \subseteq K^*$ and $F \subseteq H$ be subfields with $K \subseteq F$, and let t and u be tuples in F as in Lemma 1.2, for $(F/K,v^*)$, such that $F/K(t,u)$ is finite and separable. Then there is an embedding $\iota : F \to K^*$ over K such that $v^*(\iota(a)) = v^*(a)$ and $\overline{\iota(a)} = \overline{a}$ for all $a \in F$, the latter whenever $v^*(a) \geq 0$.*

Proof. Write v for v^*. By density H/K^* is immediate, so the statement makes sense, and if v is trivial then $H = K^*$. So let v be nontrivial. Then H is henselian, being algebraic over K^*. Take $b \in \mathcal{O}_F$ with $F = K(t,u)(b)$, the minimal polynomial g of b over $K(t,u)$, and $e \in K^{*\times}$ with $eg \in \mathcal{O}_H[Z]$. Put $\delta := v(eg'(b)) < \infty$. For t',u' in K^* close to t,u, which exist by density, we have $v(t_i') = v(t_i)$ and $\overline{u_j'} = \overline{u_j}$. So, as in the proof of Lemma 4.1, $t \mapsto t'$, $u \mapsto u'$ defines a K-isomorphism $\iota : K(t,u) \to K(t',u')$ that preserves values and residues. The coefficients of $f := e\,\iota(g)$ are those of eg, as rational functions over K, evaluated at (t',u') instead of (t,u). So for (t',u') close enough to (t,u), we have $f \in \mathcal{O}_H[Z]$, $v(f'(b)) = \delta$ and $v(f(b)) = v(f(b) - eg(b)) > 2\delta$. Then Newton's Lemma [EP05, Theorem 4.1.3(5)] in H gives a root $b' \in \mathcal{O}_H$ of f with $v(b' - b) > \delta$. Taylor expansion at b' gives $f(b) = (b - b')\big(f'(b') + (b - b')c\big)$ with $c \in \mathcal{O}_H$, so $v(b - b') = v(f(b)) - \delta$, which is large when (t',u') is close to (t,u). The element b' is separably algebraic over K^*, being a root of $\iota(g)$, and lies in H, so $b' \in K^*$. So ι extends to F by $b \mapsto b'$.

The valuation rings of F over $\mathcal{O}_{K(t,u)}$ are finitely many, say $\mathcal{O}_F = \mathcal{O}_1,\dots,\mathcal{O}_m$ [EP05, Theorem 3.3.4], and pairwise incomparable [EP05, Lemma 3.2.8]. Choose $a_\ell \in \mathcal{O}_1 \setminus \mathcal{O}_\ell$ for $\ell \geq 2$, and $a_1',\dots,a_q' \in \mathcal{O}_F$ whose residues

generate k_F over $k_{K(t,u)}$ [EP05, Corollary 3.2.3]. As $\iota(R(t,u,b)) = R(t',u',b')$ for rational functions R over K, we may choose (t',u') so close to (t,u) that $v(\iota(a)-a) > 0$ for all these a. Then the valuation ring of $v \circ \iota$ lies over $\mathcal{O}_{K(t,u)}$ and contains $a_2, \dots, a_m$, so it is $\mathcal{O}_F$. The induced map of value groups is the identity on $\Gamma_{K(t,u)}$, hence on Γ_F, as $\Gamma_F/\Gamma_{K(t,u)}$ is torsion [EP05, Corollary 3.2.3] and Γ_{K^*} is torsion-free. The induced map of residue fields is the identity on $k_{K(t,u)}$ and on the $\overline{a'_j}$, hence on k_F. □

Proposition 4.3 (Embedding extensions without transcendence defect, cf. [KP16, §3.2 and Lemma 3.7]). *Let (K,v) be a separably tame field, and let $(L/K,v)$ be an extension with Γ_L/Γ_K torsion-free and k_L/k_K separable. Let T be a family in L as in Lemma 1.2. If $\mathrm{char}K = 0$, suppose that L is separably algebraic over $K(T)$. If $\mathrm{char}K = p$, suppose that all p^n-th roots of elements of T lie in L, and that L is separably algebraic over the field $K(T^{1/p^\infty})$ they generate. Let (K^*,v^*) be a separably tame extension of (K,v) with v^* nontrivial, and suppose that (K^*,v^*) is $|L|^+$-saturated. Let $\rho\colon \Gamma_L \to \Gamma_{K^*}$ and $\sigma\colon k_L \to k_{K^*}$ be embeddings over Γ_K and k_K, respectively. Then there is an embedding $\iota\colon (L,v) \to (K^*,v^*)$ over K that respects ρ and σ.*

Proof. By **(S)**, it suffices to embed each finitely generated subextension F of L/K respecting ρ and σ. Its generators have separable minimal polynomials over $K(T)$, resp. $K(T^{1/p^\infty})$, with finitely many coefficients. So F is separably algebraic over $K(T_0)$ for a finite tuple T_0 consisting of elements of T, resp. of the p^n-th roots of elements of T for a single n. The entries of T_0 are again as in Lemma 1.2, as p^n-th roots divide values by p^n and take p^n-th roots of residues. Enlarging F, we may assume that T_0 lies in F, so that $F/K(T_0)$ is finite and separable, and $(F/K,v)$ is without transcendence defect. If $\mathrm{char}K = 0$, then (K,v) is tame, hence defectless, as tame extensions have trivial defect. So $(F^{\mathrm{h}}/K,v)$ is strongly inertially generated by Proposition 2.31**(i)** and the remark after it, and Lemma 4.1 applies.

So let $\mathrm{char}K = p$, and put $K' := K^{\mathrm{perf}}$, $F' := FK'$ and $H := (K^*)^{\mathrm{perf}}$. By Lemma 1.9, K' is tame, hence defectless, and its residue field is perfect: it is k_K if v is nontrivial on K, and K' otherwise. Also K^* is dense in H, Γ_{K^*} is p-divisible and k_{K^*} is perfect. As K'/K is algebraic, $\mathrm{trdeg}(F'/K') = \mathrm{trdeg}(F/K)$, while the rational rank of the value group extension and the transcendence degree of the residue field extension can only grow. So $(F'/K',v)$ is without transcendence defect, by the Abhyankar Inequality. Moreover $\Gamma_{F'} \subseteq \frac{1}{p^\infty}\Gamma_F$, and $\Gamma_{F'}/\Gamma_{K'}$ is torsion-free. This is clear if $\Gamma_{K'} = \Gamma_K = 0$. Otherwise $\Gamma_{K'} = \Gamma_K$ is p-divisible, and if $N\gamma \in \Gamma_K$ and $p^m\gamma \in \Gamma_F$, then $p^m\gamma \in \Gamma_K$, so $\gamma \in \Gamma_K$. By Proposition 2.31**(i)** and the remark after it, $(F'^{\mathrm{h}}/K',v)$ is strongly inertially generated. The embeddings $\rho|_{\Gamma_F}$ and $\sigma|_{k_F}$ extend uniquely to $\frac{1}{p^\infty}\Gamma_F \supseteq \Gamma_{F'}$ and $k_F^{1/p^\infty} \supseteq k_{F'}$, with values in $\Gamma_{K^*} = \Gamma_H$ and $k_{K^*} = k_H$. As H is henselian, Lemma 4.1 gives an embedding $j\colon F'^{\mathrm{h}} \to H$ over K' that respects these extensions. Then $j(F)$ is finite and separable over $K(j(T_0))$, and the entries of $j(T_0)$ are as in Lemma 1.2, as j respects ρ and σ. So Lemma 4.2 gives an embedding $\iota_1\colon j(F) \to K^*$ over K that preserves values and residues. The embedding $\iota_1 \circ j|_F$ respects ρ and σ. □

4.2. **Immediate extensions.**

Lemma 4.4 (cf. [Kuh16, Lemma 6.2]). *Let (M,v) be a valued field, and let z be a limit of a pseudo-Cauchy sequence $(c_\alpha)_{\alpha<\lambda}$ in M of transcendental type, in some valued field extension of M. Let (K^*,v^*) be a henselian valued field that is $|M|^+$-saturated, and let $\iota\colon (M,v) \to (K^*,v^*)$ be an embedding. Then there is a nonempty open subset U of K^* such that for every $z' \in U$ there is an embedding of $(M(z)^{\mathrm{h}},v)$ into (K^*,v^*) that extends ι and sends z to z'.*

Proof. Put $c'_\alpha := \iota(c_\alpha)$ and $\gamma'_\alpha := v^*(c'_{\alpha+1} - c'_\alpha)$. Any finitely many of the formulas $y \neq 0$, $v(x - c'_{\alpha+1}) > \gamma'_\alpha$ and $v(y) > \gamma'_\alpha$, for $\alpha < \lambda$, hold for $x = c'_{\beta+1}$ and $y = c'_{\beta+2} - c'_{\beta+1}$ with β large. By saturation, some $x_0, y_0 \in K^*$ satisfy all of them. Let $U := \{z' \in K^* \mid v^*(z'-x_0) > v^*(y_0)\}$. Every $z' \in U$ satisfies $v^*(z' - c'_{\alpha+1}) > \gamma'_\alpha$, hence $v^*(z'-c'_\alpha) = \gamma'_\alpha$, for all α. So z' is a limit of $(c'_\alpha)_\alpha$, which is a pseudo-Cauchy sequence in $\iota(M)$ of transcendental type. Transporting $M(z)$ along ι and applying **(K2)** over $\iota(M)$, we see that ι extends to $M(z) \to \iota(M)(z')$ with $z \mapsto z'$, and then to $M(z)^{\mathrm{h}}$ [EP05, Theorem 5.2.2]. □

Proposition 4.5 (Immediate extensions, cf. [Kuh16, Theorem 1.7(b)], [KP16, Theorem 4.2]). *Let (M,v) be separably tame, let $(L/M,v)$ be an immediate extension such that L/M is separable, and let (K^*,v^*) be a henselian valued field that is $|L|^+$-saturated. Then every embedding $\iota\colon (M,v) \to (K^*,v^*)$ extends to an embedding of (L,v). Suppose moreover that $\mathrm{char}M = p$, that ι is separable, and that $C \subseteq L$ is such that $B \cup C$ is p-independent in L for a p-basis B of M, with $|C| \leq \mathrm{imp}(K^*/\iota(M))$. Then the extension can be chosen so that $\iota(B \cup C)$ is p-independent in K^*.*

Proof. If v is trivial on M, then immediacy forces $L = M$, and then $C = \emptyset$. So let v be nontrivial on M. Being p-independent is expressed by first-order formulas. So by **(S)**, it suffices to extend ι to each finitely

generated subextension F of L/M, such that $\iota(B \cup C_0)$ is p-independent for a given finite $C_0 \subseteq C \cap F$. If $\mathrm{char}M = 0$, put $C_0 := \emptyset$. Let $N := F^{\mathrm{h}}$ if $\mathrm{char}k_M = 0$. Otherwise let N be a separably tame immediate separably algebraic extension of F^{h}, which exists by Proposition 3.25. In both cases N is separably tame, and $(N/M, v)$ is immediate and separable. In characteristic p, $F/M(C_0)$ is separable and C_0 is algebraically independent over M, by **(P1)** and **(P3)**: an element of C_0 algebraic over M and the others would be separably algebraic over them, hence in the subfield generated by them and F^p. So there is a separating transcendence basis $x_1, \dots, x_n$ of F/M that begins with the $k := |C_0|$ elements of C_0 [ML39]. Let M_i be the relative algebraic closure of $M(x_1, \dots, x_i)$ in N. The extension $N/M(x_1, \dots, x_n)$ is separably algebraic, since $F/M(x_1, \dots, x_n)$, F^{h}/F and N/F^{h} are. As $M(x_1, \dots, x_n)/M(x_1, \dots, x_i)$ is purely transcendental, $N/M(x_1, \dots, x_i)$ is separable, and hence so is N/M_i, by **(P4)**. So N/M_i is regular, M_i is separably tame by Lemma 1.10, and $B \cup \{x_1, \dots, x_i\}$ is a p-basis of M_i by **(P3)** and **(P2)**. Also $M_0 = M$, as M has no proper immediate separably algebraic extension, and $F \subseteq M_n$.

We extend ι to $M_1, \dots, M_n$ in turn, such that $\iota(x_{i+1}) \notin K^{*p}(S_i)$ for $i < k$, where $S_i := \iota(B \cup \{x_1, \dots, x_i\})$. As $|M_n| \leq |N| \leq |L|$, by **(S)** it suffices to extend ι from M_i to each finitely generated subextension G of M_{i+1}/M_i that contains x_{i+1}. Such $(G/M_i, v)$ is immediate and separable, of transcendence degree 1. So $G^{\mathrm{h}} = M_i(y)^{\mathrm{h}}$ for some $y \in G$: by Proposition 3.23 if $\mathrm{char}k_M = p$, and for any $y \in G \setminus M_i$ if $\mathrm{char}k_M = 0$, as $G^{\mathrm{h}}/M_i(y)^{\mathrm{h}}$ is then finite, immediate and defectless by Ostrowski's lemma. By Lemmas 1.14**(i)** and 4.4, there is a nonempty open subset U of K^* such that for every $y' \in U$, ι extends to an embedding $j : M_i(y)^{\mathrm{h}} \to K^*$ with $j(y) = y'$. If $i \geq k$, any $y' \in U$ will do. If $i < k$, then $K^{*p}(S_i) \neq K^*$, as $|S_i \setminus \iota(B)| = i < k \leq \mathrm{imp}(K^*/\iota(M))$. A proper subfield E of K^* does not contain U ("wiggling", cf. [vdS25, SM25]). Otherwise E contains $U - U$, hence $W := \{w \mid v^*(w) > \delta\}$ for some δ, and each $a \in K^{*\times}$ is $(aw)w^{-1}$ with $aw, w \in W$. So we may take $y' \notin K^{*p}(S_i)$. Then j is separable, as it maps the p-basis $B \cup \{x_1, \dots, x_i, y\}$ of $M_i(y)^{\mathrm{h}}$, given by **(P3)** and **(P2)**, to $S_i \cup \{y'\}$. As $B \cup \{x_1, \dots, x_{i+1}\}$ is p-independent in N, hence in $M_i(y)^{\mathrm{h}}$, its image $S_i \cup \{j(x_{i+1})\}$ is p-independent, i.e. $j(x_{i+1}) \notin K^{*p}(S_i)$. At the end, $\iota(B \cup C_0) = S_k$ is p-independent. □

Kuhlmann and Pal deduce the first part of Proposition 4.5 from the tame case, through perfect hulls [Kuh16, §7.2]. The ambient field N lets us apply Henselian rationality over separably tame fields directly.

4.3. **The Lambda relative embedding property.** Let $\mathfrak{L}$ be an expansion of $\mathfrak{L}_{\mathrm{val}}$, and let $\mathbf{C}$ be a class of $\mathfrak{L}$-structures expanding valued fields. The following property is from [Ans25, §4.1].

Definition 4.6 (Lambda relative embedding property)**.** We say that $\mathbf{C}$ has the **Lambda relative embedding property** (ΛREP) if for all $K_1, K_2 \in \mathbf{C}$ that extend a separably tame valued field K for which

- **(i)** K_1/K and K_2/K are separable,
- **(ii)** $\mathrm{imp}(K_1/K) \leq \mathrm{imp}(K_2/K)$,
- **(iii)** K_2 is $|K_1|^+$-saturated,
- **(iv)** Γ_{K_1}/Γ_K is torsion-free and k_{K_1}/k_K is separable, and
- **(v)** $\rho : \Gamma_{K_1} \to \Gamma_{K_2}$ and $\sigma : k_{K_1} \to k_{K_2}$ are embeddings over Γ_K and k_K respectively, and σ is separable,

there exists a separable embedding $\iota : K_1 \to K_2$ over K inducing ρ and σ.

Remark 4.7. The separability of k_{K_1}/k_K in **(iv)** is automatic if v is nontrivial on K, since then k_K is perfect. Likewise σ is separable whenever v is nontrivial on K_1, since then k_{K_1} is perfect. This hypothesis on σ cannot be dropped when trivially valued fields are allowed. For example, let $K := \mathbb{F}_p$ and $K_1 := \mathbb{F}_p(x)$, let K_2 be an $\aleph_1$-saturated elementary extension of $\mathbb{F}_p(x)$, all trivially valued, and let $\sigma(x) := x^p$. Then **(i)**–**(iv)** hold, but the only embedding inducing σ is σ itself, and K_2 is not separable over $\mathbb{F}_p(x^p)$. If $\mathrm{imp}(K) = \mathrm{imp}(K_1) = \mathrm{imp}(K_2) < \infty$, then **(i)** implies **(ii)**, and every K-embedding of K_1 into K_2 is separable: a p-basis of K is then a p-basis of K_1, by **(P1)**.

Theorem 4.8 (cf. [KP16, Theorem 5.1] for fields of a fixed finite imperfection degree, and [Ans25, Theorem 4.19] in general)**.** **Mod(STVF)** *has the* Λ*REP.*

We follow [KP16, proof of Lemma 4.3] for the part without transcendence defect, and [Ans25, proof of Theorem 4.19] for the p-basis.

Proof. Let K, K_1, K_2, ρ and σ be as in Definition 4.6, with $K_1, K_2 \models$ **STVF**, and write v for all three valuations. In characteristic 0, conditions **(i,ii)** are void, Step 1 is not needed, and Steps 2 and 3 need no p-th roots and no p-basis. Furthermore in residue characteristic 0 the separability conditions in **(iv,v)** are void.

First we deal with the case of trivial valuations. If v is trivial on K_2, then it is trivial on K_1, as ρ is injective. Then the residue maps identify the fields with their residue fields, and $\iota := \sigma$ is as required. Suppose that v is trivial on K_1 but not on K_2. A finitely generated subextension F of K_1/K has a separating transcendence basis

u [ML39]. As v is trivial on F, $F/K(u)$ is inert, so Lemma 4.1 embeds F respecting ρ and σ. By **(S)**, so does K_1. The embedding ι is separable by **(P5)** and **(P1)**: for a p-basis P of K_1, the units $\iota(P)$ have residues $\sigma(P)$, which are p-independent as σ is separable. So we may suppose from now on that v is nontrivial on K_1 and K_2.

Step 1: reduction to $\aleph_0$-saturated K_1. Let $\mathfrak{K} = (K, K_1, K_2, \rho, \sigma)$ be the structure consisting of K_1, K_2, their common subfield K, ρ and σ. Let $\mathfrak{K}^* = (K^*, K_1^*, K_2^*, \rho^*, \sigma^*)$ be an $\aleph_0$-saturated elementary extension, and let $K_2^{**} \succeq K_2^*$ be $|K_1^*|^+$-saturated. These data satisfy the hypotheses of Definition 4.6. Indeed, separable tameness, **STVF** and **(i)**, **(iv)**, **(v)** are first-order in $\mathfrak{K}$, for **(i)** by **(P1)**. And K_2^{**}/K_2^* is separable, as a p-dependence in K_2^{**} between elements of K_2^* is witnessed in K_2^*. For **(ii)**, if K_2 has finite p-degree, then so do K and K_1, and **(ii)** compares differences of p-degrees, which are first-order. If K_2 has infinite p-degree, let B^* be a p-basis of K^*. By saturation, for every $S \subseteq K_2^{**}$ with $|S| \leq |K_1^*|$ some element lies outside $K_2^{**p}(B^* \cup S)$. So $\operatorname{imp}(K_2^{**}/K^*) > |K_1^*| \geq \operatorname{imp}(K_1^*/K^*)$. Now let $\iota^* : K_1^* \to K_2^{**}$ be as required. Its restriction to K_1 respects ρ and σ, and it is separable, as K_1^*/K_1 is separable for the same reason as K_2^{**}/K_2^*. By **(P1)**, such embeddings of K_1 are described by a set Σ of formulas with at most $|K_1|$ parameters in K_2, as in **(S)**, together with the formulas saying that $x_{b_1}, \dots, x_{b_m}$ are p-independent, for p-independent $b_1, \dots, b_m \in K_1$. As Σ is realized in $K_2^{**} \succeq K_2$, it is finitely satisfiable in K_2, hence realized in K_2.

Step 2: the part without transcendence defect. Choose a maximal family $(\gamma_i)_{i\in I}$ in Γ_{K_1} rationally independent over Γ_K, and a transcendence basis $(\xi_j)_{j\in J}$ of k_{K_1}/k_K. As Γ_{K_1} is p-divisible and k_{K_1} is perfect (Lemma 1.8), $\aleph_0$-saturation gives $t_i, u_j \in \bigcap_n K_1^{p^n}$ with $v(t_i) = \gamma_i$, $v(u_j) = 0$ and $\overline{u_j} = \xi_j$. Let T be the family of these elements, so that $K' := K(T^{1/p^\infty}) \subseteq K_1$, and let K_0 be the relative algebraic closure of K' in K_1. Let B be a p-basis of K. As $K' = K'^p(B)$, and B is p-independent in K_1 by **(P1)**, B is a p-basis of K' and K_1/K' is separable. By **(P4)** and **(P2)**, K_1/K_0 is separable, K_0/K' is separably algebraic, and B is a p-basis of K_0. By Lemma 1.2, $\Gamma_{K_1}/\Gamma_{K_0}$ is torsion and k_{K_1}/k_{K_0} is algebraic. As K_1/K_0 is regular, Lemma 1.10 shows that K_0 is separably tame, that $\Gamma_{K_1}/\Gamma_{K_0}$ is torsion-free, hence trivial, and that $k_{K_1} = k_{K_0}$, as v is nontrivial on K_0. So $(K_1/K_0, v)$ is immediate. By **(iv)** and Proposition 4.3, there is an embedding $\iota_0 : (K_0, v) \to (K_2, v)$ over K that respects ρ and σ. It is separable by **(P1)**, as $\iota_0(B) = B$ is p-independent in K_2.

Step 3: the immediate part. By **(P1)**, there is $C \subseteq K_1$ such that $B \cup C$ is a p-basis of K_1, and $|C| = \operatorname{imp}(K_1/K)$. As B is a p-basis of $\iota_0(K_0)$, we have $\operatorname{imp}(K_2/\iota_0(K_0)) = \operatorname{imp}(K_2/K) \geq |C|$, by **(ii)**. By Proposition 4.5, ι_0 extends to an embedding $\iota : (K_1, v) \to (K_2, v)$ with $\iota(B \cup C)$ p-independent. So ι is separable, by **(P1)**. It respects ρ and σ, as ι_0 does and $(K_1/K_0, v)$ is immediate: each $a \in K_1^\times$ has some $a_0 \in K_0$ with $v(a - a_0) > v(a)$. □

Remark 4.9 (The base field in ΛREP)**.** The proof of Theorem 4.8 uses that K is separably tame in two places only. In Step 1 it is part of the first-order data transferred to $\mathfrak{K}^*$. In Step 2 it enters through Proposition 4.3, which needs a defectless field over which Proposition 2.31 applies, namely K^{perf} (Lemma 1.9). Step 3 applies Proposition 4.5 over K_0, which is separably tame by Lemma 1.10 whatever K is.

- **(i)** The property in Definition 4.6 does not change if "K is separably tame" is weakened to "K^{h} is separably tame". Indeed, K^{h} lies in K_1 and in K_2, as both are henselian. As K^{h}/K is immediate and separably algebraic, **(i)**, **(ii)**, **(iv)** and **(v)** hold for K^{h} in place of K, by **(P2)** and **(P4)**. And an embedding of K_1 over K maps $K^{\mathrm{h}} \subseteq K_1$ onto $K^{\mathrm{h}} \subseteq K_2$, by the universal property of the henselization [EP05, Theorem 5.2.2].
- **(ii)** Kuhlmann's relative embedding property [Kuh16, §6] asks instead that the base K be defectless. Under **(iv)**, with K_1 separably tame, this implies that K^{h} is separably tame, so it is a special case of **(i)**. Indeed, if v is trivial on K, then K is separably tame. Otherwise Γ_{K_1} is p-divisible and k_{K_1} is perfect, by Lemma 1.8. So Γ_K is p-divisible, as Γ_{K_1}/Γ_K is torsion-free, and k_K is perfect, as k_{K_1}/k_K is separable. And K^{h} is defectless by **(D1)**, so it has no proper finite immediate extension. By Lemma 1.8, K^{h} is separably tame. Conversely, a separably tame field that is imperfect and nontrivially valued is not defectless (Section 4.1). So Definition 4.6 allows strictly more base fields than [Kuh16, §6]. Under **(iv)**, the hypothesis "K^{h} is separably tame" amounts to "K^{h} is separably algebraically maximal", by the same argument.
- **(iii)** Some hypothesis on K is needed: it does not suffice that K be henselian and perfect, with Γ_K p-divisible and k_K perfect. Let $L := \mathbb{F}_p(\!(t)\!)^{\mathrm{perf}}$ with the t-adic valuation, which is henselian, and let $K \subseteq L$ be the henselization of $\mathbb{F}_p(t)^{\mathrm{perf}}$. Then K is perfect, $\Gamma_K = \mathbb{Z}[1/p]$ and $k_K = \mathbb{F}_p$. In $\mathbb{F}_{p^p}(\!(t^{\mathbb{Z}[1/p]})\!)$, the element $\alpha := \sum_{i\geq 1} t^{-1/p^i}$ satisfies $\wp(\alpha) = 1/t$, and $\beta := \alpha + c$, with $c \in \mathbb{F}_{p^p}$ and $\wp(c) = 1$, satisfies $\wp(\beta) = 1/t + 1$. For $a \in \{1/t, 1/t+1\}$ we have $v(a - \wp(x)) < 0$ for all $x \in L$. Otherwise $v(\wp(x - \alpha)) \geq 0$ or $v(\wp(x - \beta)) \geq 0$, hence $v(x - \alpha) \geq 0$, as $v(c) = 0$ and $v(\wp(y)) = pv(y)$ when $v(y) < 0$. Then x and α have the same terms of negative exponent, which is impossible: α has infinitely many exponents in $[-1, 0)$, and an element of L only finitely many. By Lemma 1.6, $K(\alpha)/K$ and $K(\beta)/K$ are immediate of degree p. Let K_1 and K_2' be maximal immediate algebraic extensions of $K(\alpha)$ and $K(\beta)$, which exist by

Zorn's Lemma. They are algebraically maximal, hence henselian, with value group $\mathbb{Z}[1/p]$ and residue field $\mathbb{F}_p$. So they are separably tame by Lemma 1.8, and tame, being algebraic over the perfect field K. No root y of $Z^p - Z - 1/t$ lies in K_2'. Otherwise $\wp(y-\beta) = -1$, so $v(y-\beta) = 0$ and $\overline{y-\beta}$ would be a root of $Z^p - Z + 1$ in $\mathbb{F}_p$. Let $K_2 \succeq K_2'$ be $|K_1|^+$-saturated. Then K, K_1, K_2 satisfy **(i)**–**(v)**, with ρ and σ the identity maps. But K_1 does not embed in K_2 over K, since $\alpha \in K_1$ and $K_2 \models \neg\exists y\,(y^p - y = 1/t)$. Here K is not separably algebraically maximal. The same fields show that $K_1 \not\equiv_K K_2'$, although $\Gamma_{K_1} = \Gamma_{K_2'} = \Gamma_K$ and $k_{K_1} = k_{K_2'} = k_K$.

Appendix A. Frobenius-closed bases of function fields

In this appendix, $p > 0$ is a prime number, k is a perfect field of characteristic p, and l/k is an **algebraic function field**, i.e. k is relatively algebraically closed in l and $\operatorname{trdeg}(l/k) = 1$. Nothing here concerns valued fields in the sense of the body of the paper: the places used below are the places of l/k, and the result is applied in the proof of Proposition 2.14 with $l = k_F$ and $k = k_K$.

The results of this appendix are due to Kuhlmann [Kuh06, §5] (see also [Kuh, §12.3]), and we follow his proofs. Lemma A.2 is [Kuh06, Lemma 29], Lemma A.3 is contained in the proof of [Kuh06, Proposition 30], Lemmas A.4–A.6 and Theorem A.7 are his proof of [Kuh06, Theorem 10], and Lemma A.8 is [Kuh06, Lemma 25].

We write $\varphi : l \to l$ for the Frobenius $x \mapsto x^p$, and for $S \subseteq l$ we write $S^p := \varphi(S)$. Note that $\varphi(cx) = c^p\varphi(x)$, so φ carries k-subspaces to k-subspaces.

Definition A.1. A k-basis $\mathcal{B}$ of l with $1 \in \mathcal{B}$ is

- **Frobenius closed** if $\mathcal{B}^p \subseteq \mathcal{B}$;
- **freely Frobenius generated** by $E \subseteq l$ if

$$\mathcal{B} = \{1\} \cup \{\varepsilon^{p^n} \mid \varepsilon \in E,\ n \in \mathbb{N}\}$$

and the elements listed are pairwise distinct.

Lemma A.2. *We have* $\bigcap_{n\in\mathbb{N}} l^{p^n} = k$.

Proof. Let u lie in the intersection. If u is algebraic over k then $u \in k$, since k is relatively algebraically closed in l. Suppose u is transcendental over k. For each n there is $w_n \in l$ with $w_n^{p^n} = u$. As $u \notin k(u)^p$, the polynomial $X^{p^n} - u$ is irreducible over $k(u)$, so $[k(u^{1/p^n}) : k(u)] = p^n$. These fields all lie in l, which is finite over $k(u)$, a contradiction. □

Lemma A.3. *Let $\mathcal{B}$ be a k-basis of l with $1 \in \mathcal{B}$. Then $\mathcal{B}$ is Frobenius closed if and only if it is freely Frobenius generated, necessarily by $E := \mathcal{B} \setminus \big(\{1\} \cup \mathcal{B}^p\big)$.*

Proof. A freely generated basis is closed, since $(\varepsilon^{p^n})^p = \varepsilon^{p^{n+1}}$ and $1^p = 1$. Conversely, suppose $\mathcal{B}^p \subseteq \mathcal{B}$. As φ is injective, $\mathcal{B}$ is a disjoint union of φ-orbits. If $b \in \mathcal{B}$ satisfies $b^{p^n} = b$ for some $n \geq 1$, then $b^{p^n-1} = 1$, so b is a root of unity and hence algebraic over $\mathbb{F}_p \subseteq k$. Therefore $b \in k$, and since $1 \in \mathcal{B}$ and $\mathcal{B}$ is k-linearly independent, $b = 1$. If $b \in \mathcal{B}^{p^n}$ for every n, then $b \in \bigcap_n l^{p^n} = k$ by Lemma A.2, so again $b = 1$. Hence every $b \in \mathcal{B} \setminus \{1\}$ lies in a orbit $\{\varepsilon^{p^n} \mid n \in \mathbb{N}\}$ with $\varepsilon \in E$ uniquely determined, and these elements are pairwise distinct. □

By a **place** of l/k we mean a place of l that is trivial on k. We write v_P for the associated valuation, lP for the residue field, and $\deg P := [lP : k]$, which is finite. Divisors are written additively, $\deg\big(\sum_P n_P P\big) := \sum_P n_P \deg P$, and for a divisor A we put

$$\mathcal{L}(A) := \{f \in l \mid v_P(f) \geq -v_P(A) \text{ for all } P\}, \qquad \ell(A) := \dim_k \mathcal{L}(A).$$

We use the Riemann–Roch theorem in the following form (see e.g. [Sti09, Thm. 1.5.15]): there is $g \in \mathbb{N}$, the genus of l/k, with $\ell(A) = \deg A - g + 1$ whenever $\deg A > 2g - 2$. Since k is relatively algebraically closed in l we have $\mathcal{L}(0) = k$.

Fix once and for all a place P_∞ of l/k, and set

$$R := \{f \in l \mid v_P(f) \geq 0 \text{ for all } P \neq P_\infty\} = \bigcup_{n\in\mathbb{N}} \mathcal{L}(nP_\infty).$$

Lemma A.4. *Let $P \neq P_\infty$ be a place of l/k. Then:*

(i) *there is $t_P \in l$ with $v_P(t_P) = -1$ and $v_Q(t_P) \geq 0$ for all $Q \neq P, P_\infty$;*

(ii) *there are $u_{P,1}, \ldots, u_{P,d} \in R$, where $d := \deg P$, whose residues at P form a k-basis of lP.*

Proof. Choose n with $n \deg P_\infty > 2g-2+\deg P$. For **(i)**, Riemann–Roch gives $\ell(P+nP_\infty) = \deg P + n\deg P_\infty - g + 1 > \ell(nP_\infty)$, so we may take $t_P \in \mathcal{L}(P+nP_\infty)\setminus\mathcal{L}(nP_\infty)$. For **(ii)**, the k-linear map $\mathcal{L}(nP_\infty) \to lP$, $f \mapsto \overline{f}$, has kernel $\mathcal{L}(nP_\infty - P)$, and $\ell(nP_\infty) - \ell(nP_\infty - P) = \deg P = \dim_k lP$, so it is surjective. Lift a k-basis of lP. □

For $P \neq P_\infty$ and $n \in \mathbb{N}$ put $M_{P,n} := \{f \in l \mid v_P(f) \geq -n \text{ and } v_Q(f) \geq 0 \ \forall Q \neq P, P_\infty\}$, and $M_P := \bigcup_n M_{P,n}$. Note that $M_{P,0} = R$. For $n \geq 1$ write $n = p^{v(n)}m$ with $p \nmid m$.

Lemma A.5. *The elements $u_{P,i}^{p^{v(n)}} t_P^n$, for $P \neq P_\infty$, $n \geq 1$ and $1 \leq i \leq \deg P$, form a k-basis of l modulo R, and the set of these elements is closed under φ.*

Proof. Closure under φ: $\left(u_{P,i}^{p^{v(n)}} t_P^n\right)^p = u_{P,i}^{p^{v(pn)}} t_P^{pn}$, since $v(pn) = v(n)+1$.

Fix P and $n \geq 1$. The map $M_{P,n} \to lP$, $f \mapsto \overline{f t_P^{-n}}$, is k-linear with kernel $M_{P,n-1}$. Since lP is a finite extension of the perfect field k, it is perfect, so $x \mapsto x^{p^\nu}$ is a bijection of lP. Being semilinear over the automorphism $c \mapsto c^{p^\nu}$ of k, it carries k-bases to k-bases. Hence $\overline{u_{P,1}}^{p^{v(n)}}, \dots, \overline{u_{P,d}}^{p^{v(n)}}$ is a k-basis of lP, and $u_{P,i}^{p^{v(n)}} t_P^n \in M_{P,n}$ maps onto it. Therefore these elements form a k-basis of $M_{P,n}$ modulo $M_{P,n-1}$, and, letting n vary, a k-basis of M_P modulo R.

It remains to see that $l = R \oplus \bigoplus_{P\neq P_\infty} \left(M_P/R\right)$. For directness, suppose $\sum_P g_P \in R$ with $g_P \in M_P$, almost all zero. Fix $Q \neq P_\infty$. Every term other than g_Q lies in the valuation ring of v_Q, as does the sum, so $v_Q(g_Q) \geq 0$ and thus $g_Q \in R$. For spanning, let $f \in l$ and induct on $\sum_{P\neq P_\infty} \max\{0, -v_P(f)\}$: if $f \notin R$, pick $P \neq P_\infty$ with $n := -v_P(f) > 0$. By the previous paragraph there is $g \in M_{P,n}$ with $v_P(f-g) > -n$, and $f-g$ has no new poles. □

Lemma A.6. *There is a k-basis of R containing 1 and closed under φ.*

Proof. Put $l_n := \mathcal{L}(nP_\infty)$, so that $l_0 = k$ and $R = \bigcup_n L_n$. We choose, for $n = 1, 2, \dots$ successively, a finite subset $T_n \subseteq l_n$ whose image is a k-basis of l_n/l_{n-1}, subject to the requirement that

$$T_{pm} \supseteq T_m^p \qquad \text{whenever } pm = n.$$

This is possible. Indeed, if $t \in T_m$ then $v_{P_\infty}(t^p) = p\, v_{P_\infty}(t) \geq -pm$ and $v_Q(t^p) \geq 0$ for $Q \neq P_\infty$, so $t^p \in l_{pm}$. Moreover, T_m^p is linearly independent modulo l_{pm-1}. For suppose $\sum_{t\in T_m} c_t t^p \in l_{pm-1}$ with $c_t \in k$ not all zero. Since k is perfect and $\operatorname{char} l = p$,

$$\Big(\sum_{t\in T_m} c_t^{1/p} t\Big)^p = \sum_{t\in T_m} c_t t^p,$$

so $v_{P_\infty}\big(\sum_t c_t^{1/p} t\big) = \frac{1}{p} v_{P_\infty}\big(\sum_t c_t t^p\big) > -m$, whence $\sum_t c_t^{1/p} t \in l_{m-1}$, contradicting the choice of T_m. So T_m^p may be extended to a set T_{pm} as required.

Now $\mathcal{B}_R := \{1\} \cup \bigcup_{n\geq 1} T_n$ is a k-basis of R, and $t \in T_m$ implies $t^p \in T_m^p \subseteq T_{pm} \subseteq \mathcal{B}_R$, so $\mathcal{B}_R$ is closed under φ. □

Theorem A.7. *Let k be a perfect field of characteristic $p > 0$ and let l/k be an algebraic function field of transcendence degree 1 in which k is relatively algebraically closed. Then l has a k-basis containing 1 which is Frobenius closed, equivalently (Lemma A.3) freely Frobenius generated.*

Proof. Take the union of the basis of Lemma A.6 and the elements of Lemma A.5. This is a k-basis of l containing 1, and it is closed under φ since each of the two parts is. □

Lemma A.8. *Let $\mathcal{B}$ be a Frobenius-closed k-basis of l. If $C \subseteq \mathcal{B}$ is finite and $c_\beta \in k^\times$ satisfy $\sum_{\beta\in C} c_\beta \beta \in l^p$, then $C \subseteq \mathcal{B}^p$.*

Proof. Write $\sum_{\beta\in C} c_\beta \beta = z^p$ and expand $z = \sum_{\delta\in D} \lambda_\delta \delta$ over $\mathcal{B}$, with $D \subseteq \mathcal{B}$ finite and $\lambda_\delta \in k^\times$. Then $z^p = \sum_{\delta\in D} \lambda_\delta^p \delta^p$. The elements δ^p lie in $\mathcal{B}$ and are pairwise distinct, and the λ_δ^p are nonzero. By uniqueness of expansions over $\mathcal{B}$ we get $C = D^p \subseteq \mathcal{B}^p$. □

Appendix B. The Fundamental Equality

The usual references give the inequality $\sum_w e(L/K,w) f(L/K,w) \leq [L : K]$ [EP05, Theorem 3.3.4] and the equality for Galois extensions [EP05, Theorem 3.3.3]. For completeness we prove the Fundamental Equality in general, with the defect defined via henselizations as in Section 1.

Proposition B.1 (Fundamental Equality). *Let (K,v) be a valued field and let L/K be a finite extension. Then*

$$[L:K] = \sum_{w \supseteq v} e(L/K,w) f(L/K,w) d(L/K,w),$$

the sum running over the prolongations w of v to L.

Proof. Fix an extension $\tilde{v}$ of v to K^{sep}, and let $D := \{\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K) \mid \tilde{v} \circ \tau = \tilde{v}\}$ be its decomposition group. Its fixed field is the henselization K^{h} of (K,v) [EP05, Section 5.2], so $D = \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{h}})$.

First suppose that L/K is separable, and let Σ be the set of K-embeddings $\sigma : L \to K^{\mathrm{sep}}$, so that $|\Sigma| = [L:K]$. Every prolongation of v to L is of the form $\tilde{v} \circ \sigma$ with $\sigma \in \Sigma$, since all prolongations of v to K^{sep} are conjugate over K [EP05, Theorem 3.2.15]. Moreover, $\tilde{v} \circ \sigma = \tilde{v} \circ \sigma'$ if and only if $\sigma' = \tau\sigma$ for some $\tau \in D$. Indeed, if $\sigma' = \tau\sigma$ with $\tau \in D$ then clearly $\tilde{v} \circ \sigma' = \tilde{v} \circ \sigma$. Conversely, extend σ and σ' to $\rho, \rho' \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Then $\tilde{v} \circ \rho$ and $\tilde{v} \circ \rho'$ both extend $\tilde{v} \circ \sigma = \tilde{v} \circ \sigma'$ from L to K^{sep}, so $\tilde{v} \circ \rho' = \tilde{v} \circ \rho \circ \mu$ for some $\mu \in \mathrm{Gal}(K^{\mathrm{sep}}/L)$, again by [EP05, Theorem 3.2.15]. Thus $\tau := \rho'\mu^{-1}\rho^{-1}$ lies in D, and $\tau\sigma = \rho'\mu^{-1}|_L = \sigma'$. Therefore the prolongations w of v to L correspond to the D-orbits in Σ. Restriction to $\sigma(L)K^{\mathrm{h}}$ is a bijection from the orbit $D\sigma$ onto the set of K^{h}-embeddings of $\sigma(L)K^{\mathrm{h}}$ into K^{sep}, so $|D\sigma| = [\sigma(L)K^{\mathrm{h}} : K^{\mathrm{h}}]$. Since $(\sigma(L)K^{\mathrm{h}}, \tilde{v})$ is the henselization of $(\sigma(L), \tilde{v})$ [EP05, Remark 5.2.4], which is isomorphic via σ to (L,w) for $w = \tilde{v} \circ \sigma$, we get $|D\sigma| = [L^{\mathrm{h}} : K^{\mathrm{h}}] = e(L/K,w)f(L/K,w)d(L/K,w)$. Summing over the orbits gives the Fundamental Equality.

In general, let L_s be the relative separable closure of K in L. Since L/L_s is purely inseparable, each prolongation of v to L_s has exactly one prolongation to L [EP05, Corollary 3.2.10]. For such a prolongation w, taking henselizations inside a fixed algebraic closure, the field $L_s^{\mathrm{h}}L$ is henselian, being algebraic over the henselian field L_s^{h}, so it contains L^{h}. Conversely, L^{h} is henselian and contains L_s, so it contains L_s^{h}. Both inclusions hold by the universal property of henselizations [EP05, Theorem 5.2.2]. Thus $L^{\mathrm{h}} = L_s^{\mathrm{h}}L$. Moreover, L is linearly disjoint from L_s^{h} over L_s, as L/L_s is purely inseparable and L_s^{h}/L_s is separable. Hence $[L^{\mathrm{h}} : L_s^{\mathrm{h}}] = [L : L_s]$. By the separable case, $[L:K] = [L:L_s][L_s:K] = \sum_w [L:L_s][L_s^{\mathrm{h}} : K^{\mathrm{h}}] = \sum_w [L^{\mathrm{h}} : K^{\mathrm{h}}]$, the sums running over the prolongations w of v to L. □

Appendix C. More about strong bases

Lemma C.1. *Let $(F/K,v)$ be an extension of valued fields of characteristic p, and let $\mathcal{B}$ be a strong basis. The following holds.*

(SB-V) *("Additive rigidity") If $B \subseteq \mathcal{B} \setminus \mathcal{B}^p$ is finite and nonempty, and $a_b \in K^\times$ are coefficients such that $v(a_b b) = 0$ for all $b \in B$, then*

$$\sum_{b \in B} a_b b \notin \wp(F) + \mathfrak{m}_F.$$

Proof. Seeking a contradiction, suppose that $s := \sum_{b\in B} a_b b = \wp(y) + z$ with $y \in F$ and $z \in \mathfrak{m}_F$. By (SB-II) there is $w \in K\langle\mathcal{B}\rangle$ with $v(y-w) > 0$. Then $\wp(y) - \wp(w) = \wp(y-w)$ has value $v(y-w) > 0$, so $s - \wp(w) \in \mathfrak{m}_F$. Write $w = \sum_{d\in D} c_d d$ with $D \subseteq \mathcal{B}$ finite and $c_d \in K$, and put $c_x := 0$ for $x \in \mathcal{B} \setminus D$. Then

$$s - \wp(w) = \sum_{b\in B} a_b b - \sum_{d\in D} c_d^p d^p + \sum_{d\in D} c_d d.$$

By (SB-III), every d^p lies in $\mathcal{B}$, and $d \mapsto d^p$ is injective, since Frobenius is. So in the expansion of $s - \wp(w)$ with respect to $\mathcal{B}$, the coefficient of $b \in B$ is $a_b + c_b$, because $b \notin \mathcal{B}^p$. For $d \in D$ with $d^p \notin B$, the coefficient of d^p is $c_{d^p} - c_d^p$. By (SB-I), every term of this expansion has value > 0. Fix $b \in B$. From $v((a_b + c_b)b) > 0$ and $v(a_b b) = 0$ we get $v(c_b b) = 0$. Inductively, suppose $v(c_{b^{p^n}} b^{p^n}) = 0$. Since $b^{p^{n+1}} \in \mathcal{B}^p$, it is not in B, so $v\big((c_{b^{p^{n+1}}} - c_{b^{p^n}}^p) b^{p^{n+1}}\big) > 0$, while $v(c_{b^{p^n}}^p b^{p^{n+1}}) = p \cdot 0 = 0$. Hence $v(c_{b^{p^{n+1}}} b^{p^{n+1}}) = 0$. Thus $b^{p^n} \in D$ for all $n \geq 0$. As D is finite, $b^{p^m} = b^{p^n}$ for some $m < n$, and injectivity of Frobenius gives $b = b^{p^{n-m}} = \big(b^{p^{n-m-1}}\big)^p \in \mathcal{B}^p$, since $b^{p^{n-m-1}} \in \mathcal{B}$ by (SB-III). This contradicts $b \notin \mathcal{B}^p$. □

Remark C.2 (The two forms of additive rigidity)**.** Lemma 2.13 and Lemma C.1 (i.e. (SB-V)) are two cases of a single statement. Let $s = a_1 + \sum_{b\in B} a_b b$ with $a_1, a_b \in K$ and $B \subseteq \mathcal{B} \setminus \mathcal{B}^p$ finite.

(a) If $s \in \wp(F) + \mathcal{O}_F$, then $v(a_b b) \geq 0$ for all $b \in B$, and $a_1 \in \wp(K) + \mathcal{O}_K$.
(b) If $s \in \wp(F) + \mathfrak{m}_F$, then $v(a_b b) > 0$ for all $b \in B$, and $a_1 \in \wp(K) + \mathfrak{m}_K$.

Lemma 2.13 is **(a)** with $B = \emptyset$, and Lemma C.1 is **(b)** with $a_1 = 0$ and $v(a_b b) = 0$ for all $b \in B$. The proofs are the same: write $s = \wp(y) + z$, approximate y by $w \in K\langle\mathcal{B}\rangle$ using (SB-II), and compare the coefficients of $s - \wp(w)$ with respect to $\mathcal{B}$, using (SB-I) and (SB-III). Since $b^p \neq 1$ for $b \neq 1$, the coefficient of 1 is $a_1 - \wp(w_1)$, where w_1 is the coefficient of 1 in w. This gives the assertion about a_1. For $b \in B$ with $v(a_b b)$ too small, the

coefficients of $b, b^p, b^{p^2}, \dots$ in w are all nonzero, which is impossible since w has finite support. This gives the assertion about B.

Lemma C.3. *Let $(E/F/K, v)$ be a valued tower such that Γ_F is cofinal in Γ_E, and suppose that $\mathcal{B}_E$ and $\mathcal{B}_F$ are strong bases of E/F and F/K, respectively. Then $\mathcal{B}_E\mathcal{B}_F = \{b_1 b_2 \mid b_1 \in \mathcal{B}_E, b_2 \in \mathcal{B}_F\}$ is a strong basis of E/K.*

Proof. Clearly $1 = 1 \cdot 1 \in \mathcal{B}_E\mathcal{B}_F \subseteq E^\times$. Also $(b_1 b_2)^p = b_1^p b_2^p \in \mathcal{B}_E\mathcal{B}_F$ by (SB-III) for $\mathcal{B}_E$ and for $\mathcal{B}_F$, which is (SB-III).

For (SB-I), first note that the map $(b_1, b_2) \mapsto b_1 b_2$ on $\mathcal{B}_E \times \mathcal{B}_F$ is injective. Indeed, if $b_1 b_2 = b_1' b_2'$ with $b_1 \neq b_1'$, then $b_2 b_1 - b_2' b_1' = 0$, which contradicts (SB-I) for $\mathcal{B}_E$ over F, since $b_2, b_2' \in F^\times$. And if $b_1 = b_1'$, then $b_2 = b_2'$. So a finite subset of $\mathcal{B}_E\mathcal{B}_F$ is $\{b_1 b_2 \mid (b_1, b_2) \in S\}$ for a finite $S \subseteq \mathcal{B}_E \times \mathcal{B}_F$. Let $a_{b_1 b_2} \in K$ for $(b_1, b_2) \in S$, let B_1 be the set of first coordinates of the elements of S, and put $c_{b_1} := \sum_{b_2} a_{b_1 b_2} b_2 \in F$, the sum over those b_2 with $(b_1, b_2) \in S$. Then

$$v\Big(\sum_{(b_1,b_2)\in S} a_{b_1 b_2} b_1 b_2\Big) = v\Big(\sum_{b_1 \in B_1} c_{b_1} b_1\Big) = \min_{b_1 \in B_1} v(c_{b_1} b_1) = \min_{(b_1,b_2)\in S} v(a_{b_1 b_2} b_1 b_2),$$

by (SB-I) for $\mathcal{B}_E$ over F, and then by (SB-I) for $\mathcal{B}_F$ over K.

For (SB-II), let $y \in E$ and $\gamma \in \Gamma_E$. By (SB-II) for $\mathcal{B}_E$, there are a finite $B_1 \subseteq \mathcal{B}_E$ and $c_{b_1} \in F$ with $v\big(y - \sum_{b_1 \in B_1} c_{b_1} b_1\big) > \gamma$. As Γ_F is cofinal in Γ_E, for each $b_1 \in B_1$ there is $\delta_{b_1} \in \Gamma_F$ with $\delta_{b_1} \geq \gamma - v(b_1)$. By (SB-II) for $\mathcal{B}_F$, there is $t_{b_1} \in K\langle \mathcal{B}_F \rangle$ with $v(c_{b_1} - t_{b_1}) > \delta_{b_1}$. Then $v\big((c_{b_1} - t_{b_1}) b_1\big) > \gamma$ for each $b_1 \in B_1$. So $v\big(y - \sum_{b_1 \in B_1} t_{b_1} b_1\big) > \gamma$, and $\sum_{b_1 \in B_1} t_{b_1} b_1$ lies in $K\langle \mathcal{B}_E\mathcal{B}_F \rangle$. □

Remark C.4. The cofinality hypothesis of Lemma C.3 holds whenever v has rank 1 and is nontrivial on F. It cannot be omitted. Let K be trivially valued, and let $E = K(t, x)$ with the valuation for which $v(t) = (0, 1)$ and $v(x) = (1, 0)$ in $\mathbb{Z}\times\mathbb{Z}$, ordered lexicographically, and the value of a polynomial is the minimum of the values of its monomials. Put $F = K(t)$, $\mathcal{B}_F = \{t^n \mid n \in \mathbb{Z}\}$ and $\mathcal{B}_E = \{x^n \mid n \in \mathbb{Z}\}$. Then $\mathcal{B}_F$ is a strong basis of F/K, and $\mathcal{B}_E$ is a strong basis of E/F. For the density of $F[x^{\pm 1}]$ in E: a nonzero $Q \in F[x]$ is $q_m x^m(1+u)$ with $q_m \in F^\times$ and $u \in xF[x]$, so $v(u) \in v(x) + \Gamma_F$, and $(1+u)^{-1} = \sum_{k\geq 0}(-u)^k$ converges. But $K\langle \mathcal{B}_E\mathcal{B}_F \rangle = K[t^{\pm 1}, x^{\pm 1}]$ is not dense in E. Indeed, for every $P = \sum_n P_n(t) x^n$ in it, (SB-I) for $\mathcal{B}_E$ over F gives $v\big((1+t)^{-1} - P\big) \leq v\big((1+t)^{-1} - P_0(t)\big)$. The right-hand side lies in Γ_F, since $(1+t)^{-1} \notin K[t^{\pm 1}]$, so it is smaller than $v(x)$.

Lemma C.5 (Base change)**.** *Continue to assume (H) and case (RT), let $\mathcal{B}$ be a strong basis of $(F/K, v)$ with $v(b) = 0$ for all $b \in \mathcal{B}$, and let K'/K be a finite extension with $k_{K'} = k_K$. Then $(FK'/K', v)$ satisfies (H), and $\mathcal{B}$ is a strong basis of $(FK'/K', v)$.*

We observe that a strong basis $\mathcal{B}$, with $v(b) = 0$ for all $b \in \mathcal{B}$, exists in case (RT) by Proposition 2.14**(ii)**.

Proof. K' is henselian, and closed under p-th roots since K is and K'/K is separable. Moreover, F, K' are linearly disjoint over K as F/K is regular, so FK'/K' is a henselized function field of transcendence degree 1 and rank 1. Since $k_{K'} = k_K$ we have $\Gamma_{FK'} = \Gamma_{K'}$ and $k_{FK'} = k_F$, so $(FK'/K', v)$ is valuation regular, without transcendence defect, and in case (RT).

Property (SB-III) concerns $\mathcal{B}$ alone. For (SB-I), let $B \subseteq \mathcal{B}$ be finite and $c_b \in K'$ not all zero. Discarding those b with $c_b = 0$, we may assume all $c_b \in K'^\times$. Since $v(b) = 0$ we have $\gamma := \min_b v(c_b) \in \Gamma_{K'}$. Choose $d \in K'^\times$ with $v(d) = \gamma$. If $v\big(\sum_b c_b b\big) > \gamma$ then $\sum_{b\in B_0} \overline{c_b d^{-1}}\,\overline{b} = 0$ with coefficients $\overline{c_b d^{-1}} \in k_{K'} = k_K$ not all zero, contradicting (SB-I) for $(F/K, v)$, which gives the k_K-linear independence of $\overline{\mathcal{B}}$ in $k_F = k_{FK'}$. For (SB-IV), the same normalisation and Lemma 2.5**(ii)**, applied over FK', reduce the hypothesis $\mathrm{rv}\big(\sum_{b\in B} c_b b\big) \in \mathrm{RV}^p_{FK'}$ to $\sum_{b\in B_0} \overline{c_b d^{-1}}\,\overline{b} \in k_F^p$ with coefficients in k_K, which is the hypothesis of (SB-IV) for $(F/K, v)$. Finally, for (SB-II), fix a K-basis $\lambda_1, \dots, \lambda_n$ of K'. Every element of FK' is $\sum_i f_i \lambda_i$ with $f_i \in F$. Approximating each f_i from $K\langle \mathcal{B} \rangle$ by (SB-II) for $(F/K, v)$ approximates $\sum_i f_i \lambda_i$ from $K'\langle \mathcal{B} \rangle$, since $v\big(\sum_i (f_i - g_i)\lambda_i\big) \geq \min_i \big(v(f_i - g_i) + v(\lambda_i)\big)$ and the values $v(\lambda_i)$ are fixed. □

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