A NOTE ON EXISTENTIALLY T-HENSELIAN FIELDS

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Abstract. A field is existentially t-henselian if it is has the same existential theory in the first-order language of rings as a field that admits a nontrivial henselian valuation. This property turns out to be equivalent to $\mathbb{Z}$-largeness, which is a property identified in previous work with Fehm, and which holds for F if and only if $tF[[t]]$ is not Diophantine in $F((t))$, without extra constants.

In this short note, we further investigate this property in order to count the number of existential theories of henselian valuations on a given field, and to find other characterizations of existential t-henselianity.

Résumé. Un corps est existentiellement t-hensélien s'il possède la même théorie existentielle, dans le langage des anneaux du premier ordre, qu'un corps admettant une valuation hensélienne non triviale. Cette propriété s'avère équivalente à "$\mathbb{Z}$-largeness", une propriété identifiée dans mes travaux antérieurs avec Fehm, et qui est vérifiée pour F si et seulement si $F[[t]]$ n'est pas diophantienne dans $F((t))$, sans constantes supplémentaires.

Dans cette brève note, nous approfondissons cette propriété afin d'énumérer les théories existentielles des valuations henséliennes sur un corps donné, et de trouver d'autres caractérisations de la propriété de t-henselianité existentielle

In this note we define a field to be "existentially t-henselian" if it is has the same existential theory in the first-order language of rings as a field that admits a nontrivial henselian valuation. This property turns up naturally in the study of existential theories of equicharacteristic henselian valuations. Perhaps more surprisingly, it turn out to be also equivalent to "$\mathbb{Z}$-largeness", another property identified in previous work with Fehm, namely [AF17], and which holds for a field F if and only if $tF[[t]]$ is definable in $F((t))$ by an existential formula in the language of rings, without parameters.

We prove several small results on existentially t-henselian fields. As an example, we are able to count the number of existential theories of henselian valuations on a given field:

Main Theorem. *Let K be any field. There are at most three existential theories $\mathrm{Th}_\exists(K, v)$ in the language of valued fields, for equicharacteristic henselian valuations v on K.*

This theorem follows from Theorem 3.6.

1. Canonical valuations

For each field K, let $\mathsf{S}_{\mathrm{val}}(K)$ be the partially ordered set[1] of valuations of K, where the ordering corresponds to the usual notions of coarsenings and refinements of valuations. More precisely, given valuations v, w on K, we say that v is **finer** than w (and that w is **coarser** than v), denoted $v \leq w$, if $\mathcal{O}_v \subseteq \mathcal{O}_w$.

We note the following properties:

- **(i)** $\mathsf{S}_{\mathrm{val}}(K)$ is directed upwards, over sets, i.e. every non-empty subset has a maximum element. In particular there is a maximum valuation, the trivial valuation, which is denoted v_K^{triv}.
- **(ii)** The set of coarsenings of a given valuation is a chain, i.e. a total order.
- **(iii)** Every chain has an infimum.

A valuation $v \in \mathsf{S}_{\mathrm{val}}(K)$ on K is **henselian** if for every finite extension L/K there is precisely one $w \in \mathsf{S}_{\mathrm{val}}(L)$ that restricts to v. Let $\mathsf{H}(K)$ be the partially ordered set of henselian valuations on K, and we write $\mathsf{H}_2(K) := \{v \in \mathsf{H}(K) \mid Kv \text{ is separably closed}\}$ and $\mathsf{H}_1(K) := \mathsf{H}(K) \setminus \mathsf{H}_2(K)$. Then

- $\mathsf{H}_2(K) < \mathsf{H}_1(K)$, i.e. every $v_1 \in \mathsf{H}_1(K)$ is finer than every $v_2 \in \mathsf{H}_2(K)$.
- $\mathsf{H}_1(K)$ is closed upwards, is a chain, and it has a minimum if $\mathsf{H}_2(K)$ is empty.
- $\mathsf{H}_2(K)$ is closed downwards, is not in general a chain, and it has a maximum if and only if it is nonempty.

The **canonical henselian valuation** v_K is the supremum of $\mathsf{H}_2(K)$. Note that if $\mathsf{H}_2(K) \neq \emptyset$, then $v_K = \max \mathsf{H}_2(K)$; whereas if $\mathsf{H}_2(K) = \emptyset$, then $v_K = \min \mathsf{H}_1(K)$. A field K itself is **henselian** if it admits a nontrivial henselian valuation, and it is **t-henselian** if some elementary extension $K^* \succeq K$ is henselian. It is a small part of the work of Jahnke and Koenigsmann, indeed it follows from Beth's Definability Theorem, that if Kv_K is not t-henselian, then v_K is $\mathfrak{L}_{\mathrm{ring}}$-definable without parameters. There are plenty of examples of t-henselian fields that are not henselian, including $\mathbb{R}$ and $\mathbb{F}_p^{\mathrm{alg}}$, and more besides. We observe also that henselianity (of a valuation) is $\mathfrak{L}_{\mathrm{val}}$-axiomatizable, but not finitely so.

A similar game can be played with p-henselian valuations, those that extend uniquely to the p-closure, as was explored by Koenigsmann, and later by Jahnke and Koenigsmann together:

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[1] This is the Riemann–Zariski space of K when suitably topologized.

Example 1.1 (p-henselian valuations, Jahnke–Koenigsmann)**.** Let $p \in \mathbb{P}$ be a prime number. For a field K, we let $\mathsf{H}^p(K)$ be the partially ordered set of p-henselian valuation rings on K. Let $\mathsf{H}^p(K)$ be the set of p-henselian valuations on K. We write $\mathsf{H}^p_1(K) = \{v \in \mathsf{H}^p(K) \mid Kv \neq Kv(p)\}$ and $\mathsf{H}^p_2(K) = \{v \in \mathsf{H}^p(K) \mid Kv = Kv(p)\}$. Then

- $\mathsf{H}^p_2(K) < \mathsf{H}^p_1(K)$, i.e. every $v_1 \in \mathsf{H}^p_2(K)$ is strictly finer than every $v_2 \in \mathsf{H}^p_1(K)$.
- $\mathsf{H}^p_1(K)$ is closed upwards, is a chain, and it has a minimum if $\mathsf{H}_2(K)$ is empty.
- $\mathsf{H}^p_2(K)$ is closed downwards, in particular under minima of chains, but is not (in general) a chain nor closed upwards.

In contrast to the situation with henselianity, p-henselianity is finitely axiomatizable. and correspondingly membership of a valuation in each of H^p_1 and H^p_2 is finitely $\mathfrak{L}_{\mathrm{val}}$-axiomatizable. The **canonical p-henselian valuation** v^p_K is the supremum of $\mathsf{H}^p_2(K)$. Note that if $\mathsf{H}^p_2(K) \neq \emptyset$, then $v^p_K = \max \mathsf{H}^p_2(K)$; whereas if $\mathsf{H}^p_2(K)$, then $v^p_K = \min \mathsf{H}^p_1(K)$. As in the henselian case, if Kv^p_K is not p-henselian then v^p_K is $\mathfrak{L}_{\mathrm{ring}}$-definable, by Beth's Definability Theorem.

Example 1.2 (Canonical tame valuation, Ketelsen–Ramello–Szewczyk)**.** Let $\mathsf{T}(K)$ be the partially ordered set of tame valuation rings on K, and let $\mathsf{T}_1(K) = \mathsf{T}(K) \cap \mathsf{H}_1(K)$ and $\mathsf{T}_2(K) = \mathsf{T}(K) \cap \mathsf{H}_2(K)$. As in the previous cases, there is an $\mathfrak{L}_{\mathrm{ring}}$-elementary class of **t-tame** fields, which are those fields k that are $\mathfrak{L}_{\mathrm{ring}}$-elementarily equivalent to a field k' admitting a nontrivial tame valuation. We denote by v^T_K the **canonical tame valuation**, which by definition is the supremum of $\mathsf{T}_2(K)$. So if $\mathsf{T}_2(K) \neq \emptyset$, then $v^T_K = \max \mathsf{T}_2(K)$, whereas if $\mathsf{T}_2(K) = \emptyset$, then $v^T_K = \min \mathsf{T}_1(K)$. Just as in the henselian case, and in contrast to the p-henselian case, tameness is not finitely axiomatizable. One case of the main theorem of [KRS24] comes from another argument by Beth's Definability Theorem applied to this context:

Theorem 1.3 ([KRS24])**.** *If Kv^T_K is not t-tame, then v^T_K is $\mathfrak{L}_{\mathrm{ring}}$-definable.*

Example 1.4. Divisible-tame valuations are those tame valuations which also have a divisible value group. The natural definition is that k is **t-divisible-tame** if there exists $k^* \equiv k$ that admits a nontrivial divisible-tame valuation. It was shown in [AJ18, Lemma 5.1] that if k is t-divisible-tame then $k \preceq k((\mathbb{Q}))$. It was also shown in [AJ18] that for each $p \in \mathbb{P} \cup \{0\}$, there exists a t-divisible-tame field k of characteristic p that is not henselian, not real closed, and not separably closed.

Similar observations hold for "canonical valuations" in many other related settings.

Remark 1.5. Many other properties of valuations that appear in the literature behave similarly. The properties henselianity, p-henselianity, tameness[2], and divisible tameness, that appear in the preceding examples, are well-behaved in several key respects. To begin with, each of these properties is $\mathfrak{L}_{\mathrm{val}}$-axiomatizable, and most are closed under composition and decomposition. Further examples of properties P for valued fields (K, v) satisfying these basic requirements are:

- v is algebraically maximal,
- v is henselian and defectless,
- v is roughly tame,
- the value group vK is roughly p-divisible,
- the value group vK is roughly divisible,
- v is defectless and a coarsening of a given valuation w.

We need not constrain our attention to $\mathfrak{L}_{\mathrm{val}}$-axiomatizability. By working in a language of bi-valued fields, we may consider structures (K, v, w) in which v coarsens w. Thus the chain H_1 is replaced by the chain of coarsenings of each w.

Example 1.6. By [AJ24, Lemma 3.4], if (K, w) is NIP and of residue characteristic $p > 0$, then w has at most one coarsening v with an imperfect residue field. Beth's Definability Theorem then implies that v is definable in (K, v, w). There is also an explicit formula known, which is due to Scanlon.

All of the above are examples of the following theorem of Ketelsen, from her thesis:

Theorem 1.7 (Ketelsen, [Ket26, Theorem 4.2.4])**.** *Let P be a property of (possibly enriched) valued fields, and P^* be a property of valued fields such that*

(i) *P is preserved under $\mathfrak{L}'_{\mathrm{val}}$-elementary equivalence, where $\mathfrak{L}'_{\mathrm{val}} = \mathfrak{L}' \cup \{\mathcal{O}\}$ for some enrichment $\mathfrak{L}' \supseteq \mathfrak{L}$.*
(ii) *P^* is first-order axiomatizable in $\mathfrak{L}_{\mathrm{val}}$.*
(iii) *For any field F and valuations u and w on F that have property P: if $u \leq w$ and $\bar{u}$ is the valuation induced by u on Fw then $(Fw, \bar{u})$ has P^*.*

Now let (K, v) be such that $v \in \mathsf{H}_1(K)$, (K, v) satisfies P and Kv is not t-henselian of type P^*. Then $\mathcal{O}_v$ is $\mathfrak{L}'$-definable.

[2]Tameness is not closed under composition, since the composition of a tame valuation of mixed characteristic $(0, p)$ with, for example, a discretely valued henselian valuation of equal characteristic zero is not tame. The closure of the class of tame valued fields under composition is the class of roughly tame fields.

2. Diophantine henselian valuation rings and valuation ideals

In this section we consider fields K and equicharacteristic henselian nontrivial valuations v on K, with corresponding valuation rings $\mathcal{O}$. The paper [FJ17] gives a survey of the results around the subject of $\mathfrak{L}_{\mathrm{ring}}$-definable henselian valuation rings. A subset of a field K is **Diophantine** if it is of the form

$$\{x \in K \mid \exists y_1, \dots \exists y_n \in K : \ f_1(x, y_1, \dots, y_n) = \dots = f_m(x, y_1, \dots, y_n) = 0\},$$

where $f_1, \dots, f_m \in \mathbb{Z}[X, Y_1, \dots, Y_m]$. Equivalently, a subset of K is Diophantine if it is defined by an existential $\mathfrak{L}_{\mathrm{ring}}$-formula. This notion of "Diophantine" disallows parameters, i.e. the polynomials f_i are over $\mathbb{Z}$.

In [AF17], the following definition was introduced.

Definition 2.1 ([AF17, Definition 3.5])**.** Let C be a class of fields.

- **(i)** We say that C has **embedded residue** if there exist $k_1, k_2 \in C$, a nontrivial valuation v on k_1, and an embedding of rings $k_1v \to k_2$. We say that a field k has **embedded residue** if $\{k' \mid k' \equiv k\}$ has embedded residue.
- **(ii)** We say that C is $\mathbb{Z}$**-large**[3] if there exist $k_1, k_2 \in C$, a subfield $E \subseteq k_1$, and a nontrivial henselian valuation v on E with $Ev \cong k_2$. We say that a field k is $\mathbb{Z}$**-large** if $\{k' \mid k' \equiv k\}$ is $\mathbb{Z}$-large.

There is an underlying duality between the existential definability of henselian valuation rings and valuation ideals, which is somewhat twisted by the appearance of henselianity in the definition of $\mathbb{Z}$-largeness. The following theorem of [AF17] characterizes (among other things) when $F[\![t]\!]$ [respectively $tF[\![t]\!]$] is Diophantine in $F(\!(t)\!)$.

Theorem 2.2 ([AF17, Theorem 1.1])**.** *Let F be a field. Then the following are equivalent.*

- **(i)** *There is an $\exists$-$\mathfrak{L}_{\mathrm{ring}}$-formula that defines $\mathcal{O}_v$ [respectively, $\mathfrak{m}_v$] in K for some equicharacteristic henselian nontrivially valued field (K, v) with residue field F.*
- **(ii)** *There is an $\exists$-$\mathfrak{L}_{\mathrm{ring}}$-formula that defines $\mathcal{O}_v$ [respectively, $\mathfrak{m}_v$] in K for every henselian valued field (K, v) with residue field elementarily equivalent to F.*
- **(iii)** *There is no elementary extension $F \preceq F^*$ with a nontrivial valuation v on F^* for which the residue field F^*v embeds into F^* [respectively, with a nontrivial henselian valuation v on a subfield E of F^* with $Ev \cong F^*$].*

Remark 2.3. By [AF17, Lemma 3.7], condition **(iii)** is equivalent to

(iii') The field F does not have embedded residue [respectively, is not $\mathbb{Z}$-large].

We give several negative examples, i.e. examples of $\mathcal{O}$ that are not Diophantine in K.

Example 2.4 (Negative examples)**.**

- **(i)** $\mathbb{C}[\![t]\!]$ is not Diophantine in $\mathbb{C}(\!(t)\!)$: this folkloric result is explained in [AK14, Appendix A].
- **(ii)** $\mathbb{Q}_p[\![t]\!]$ is not Diophantine in $\mathbb{Q}_p(\!(t)\!)$: this is also explained in [AK14, Appendix A].
- **(iii)** $\mathbb{R}[\![t]\!]$ is not Diophantine in $\mathbb{R}(\!(t)\!)$: this is a similar direct limit argument. Any existential formula defining $\mathbb{R}[\![t]\!]$ in $\mathbb{R}(\!(t)\!)$ must also define the (nontrivial) valuation ring $\mathbb{R}[\![t]\!]^{\mathrm{Px}}$ in the Puiseux series $\mathbb{R}(\!(t)\!)^{\mathrm{Px}}$, but this is a real closed field.

These examples generalize to the following.

Example 2.5. $\mathbb{F}[\![t]\!]$ is not Diophantine in $\mathbb{F}(\!(t)\!)$, for any algebraically closed field $\mathbb{F}$.

Proof. To see this, we again give a direct limit argument: in the algebraic closure of $\mathbb{F}(\!(t)\!)$ the unique prolongation of $\mathbb{F}[\![t]\!]$ is again nontrivial and must be defined by any existential formula defining $\mathbb{F}[\![t]\!]$ in $\mathbb{F}(\!(t)\!)$ since the algebraic closure is a direct limit of isomorphic copies of $\mathbb{F}(\!(t)\!)$. □

The arguments here can be easily extended to show that $F[\![t]\!]$ is not Diophantine in $F(\!(t)\!)$ whenever F is a characteristic zero t-henselian field, i.e. elementarily equivalent to one admitting a nontrivial henselian valuation. In fact, as the main theorem will show, the characteristic assumption may be removed.

Turning to positive examples, we have the following.

Example 2.6 (Positive examples)**.**

- **(i)** $\mathbb{F}_q[\![t]\!]$ is Diophantine in $\mathbb{F}_q(\!(t)\!)$ for all prime powers q ([AK14]).
- **(ii)** $F[\![t]\!]$ is Diophantine in $F(\!(t)\!)$, for F a PAC field not containing the algebraic closure of its prime subfield ([Feh15]).
- **(iii)** $\mathbb{Q}[\![t]\!]$ is Diophantine in $\mathbb{Q}(\!(t)\!)$ ([AF17]).

Each of these can be seen in a rather concrete fashion, with explicit formulas.

Remark 2.7. Strictly speaking, the framework of [AF17] was that of C-**fields**: which are fields F equipped with distinguished morphisms $C \to F$, for a given integral domain C. In the present note we are always working simply with the case $C = \mathbb{Z}$.

[3] The paper [AF17] works in the context of "C-fields", that is fields F equipped with a distinguished morphism $C \to F$, for a given integral domain C. In the present note we are always working simply in the case $C = \mathbb{Z}$. What we here call $\mathbb{Z}$-large is there simply called "large".

3. Existentially t-henselian fields

For a field K, we denote by $\mathsf{H}^e(K)$ the set of (equivalence classes of) equicharacteristic henselian valuations on K, partially ordered by the relation of refinement/coarsening, with largest element v^{triv}, the trivial valuation on K. Let E denote the $\mathfrak{L}_{\mathrm{ring}}$-fragment $\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$. Let $\mathbf{H}^{e\prime}$ denote the $\mathfrak{L}_{\mathrm{val}}$-theory of equicharacteristic henselian nontrivially valued fields, and let $\mathbf{H}^{e\prime}(R)$ denote $\mathbf{H}^{e\prime}$ together with axioms that impose that R holds on the residue field, for any $\mathfrak{L}_{\mathrm{ring}}$-theory R. The following theorem from [AF26]—though it is based on [AF16]—will be used several times in this section.

Theorem 3.1 ([AF26, Corollary 3.19(a)(III)]). *Let $(K,v)\models\mathbf{H}^{e\prime}$. Then* $\mathrm{Th}_\exists(K,v)=\mathbf{H}^{e\prime}(\mathrm{Th}(Kv))_\exists=\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_\exists$.

The actual statement of [AF26, Corollary 3.19(a)(III)] is $\mathrm{Th}_\exists(K,v)=\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_\exists$, which yields the theorem when combined with the easy inclusions $\mathrm{Th}_\exists(K,v)\supseteq\mathbf{H}^{e\prime}(\mathrm{Th}(Kv))_\exists\supseteq\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_\exists$.

Remark 3.2. In fact there is a uniform version of Theorem 3.1: for every $\mathfrak{L}_{\mathrm{ring}}$-theory R we have $\mathbf{H}^{e\prime}(R)_\exists=\mathbf{H}^{e\prime}((R\cup\mathbf{F})_\exists)_\exists$, where $\mathbf{F}$ is the $\mathfrak{L}_{\mathrm{ring}}$-theory of fields. This is not stated explicitly in [AF26], although it is used in several examples, but follows from the "in particular" statement of [AF26, Proposition 2.24], which is given in an abstract setting.

We say that F satisfies **(ð)**, or is **existentially t-henselian**[4], if it satisfies the equivalent conditions in the following lemma, which should be compared with [AF17, Section 6.3].

Lemma 3.3. *For a field F, the following are equivalent.*

(i) $\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F(\!(t)\!))$.
(ii) $\mathrm{Th}_\exists(F)=\mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$.
(iii) $\mathrm{Th}_\exists(F)=\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_{\mathrm{E}}$.
(iv) $\mathrm{Th}_\exists(F)$ *is a fixed point of the map* $T\longmapsto\mathbf{H}^{e\prime}(T)_{\mathrm{E}}$ *from the power set of* E *to itself.*
(v) *There exists a henselian field F' such that* $\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F')$.
(vi) *F is $\mathbb{Z}$-large.*
(vii) $\{F'\mid\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F')\}$ *is $\mathbb{Z}$-large.*

Proof. From Theorem 3.1 we have $\mathrm{Th}_\exists(F(\!(t)\!),v_t)=\mathbf{H}^{e\prime}(\mathrm{Th}(F))_\exists=\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_\exists$. Taking the reduct to $\mathfrak{L}_{\mathrm{ring}}$, this yields $\mathrm{Th}_\exists(F(\!(t)\!))=\mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}=\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_{\mathrm{E}}$, and thus **(i,ii,iii,iv)** are equivalent. If there exists $v\in\mathsf{H}^e(F)$ non-trivial, then $\mathrm{Th}_\exists(F,v)=\mathrm{Th}_\exists(F(\!(t)\!),v\circ v_t)$, and so $\mathrm{Th}_\exists(F)=\mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$. Next suppose that there exists henselian F' such that $\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F')$. Replacing F' with an elementary extension if necessary, we may assume that there exists $v\in\mathsf{H}^e(F')$ non-trivial. Then $\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F')=\mathbf{H}^{e\prime}(\mathrm{Th}(F'))_{\mathrm{E}}=\mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$, which shows that **(v)⇒(ii).** Conversely, if $\mathrm{Th}_\exists(F)=\mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$ then $\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F(\!(t)\!))$. Thus **(v)⇒(i).**

By [AF17, Proposition 6.10 (1 ⇔ 4)], $\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F(\!(t)\!))$ is equivalent to $\mathbb{Z}$-largeness, which proves **(i)⇔(vi).** Clearly **(vi)⇒(vii).**

Suppose that for some $F_1,F_2\in\{F'\mid\mathrm{Th}_\exists(F)=\mathrm{Th}_\exists(F')\}$ there exists $F_0\subseteq F_2$ and a non-trivial henselian valuation w on F_0 with F_0w isomorphic to F_1. Then w is equicharacteristic and henselian, so we may embed $F_1(t)^h$ into F_2. This shows that $\mathrm{Th}_\exists(F_1)\subseteq\mathrm{Th}_\exists(F_1(t)^h)\subseteq\mathrm{Th}_\exists(F_2)$. By [AF17, Proposition 6.10 (4 ⇒ 1)], we conclude that F is $\mathbb{Z}$-large, i.e. **(vii)⇒(vi).** □

Lemma 3.4. *Let K be a field and $v\in\mathsf{H}^e(K)$ non-trivial. Then Kv satisfies* **(ð)** *if and only if* $\mathrm{Th}_\exists(K,v)=\mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$.

Proof. For the implication ⇒ we have

$$\begin{aligned}\mathrm{Th}_\exists(K,v)&=\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_\exists &&\text{by Theorem 3.1}\\&=\mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_{\mathrm{E}})_\exists &&\text{by Lemma 3.3(iii) since } Kv \text{ satisfies (ð)}\\&=\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(K))_\exists &&\text{by Theorem 3.1}\\&=\mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists, &&\text{by Theorem 3.1.}\end{aligned}$$

For the converse, from the hypothesis $\mathrm{Th}_\exists(K,v)=\mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ it follows that $\mathrm{Th}_\exists(Kv)=\mathrm{Th}_\exists(K)$, and K admits a non-trivial henselian valuation, thus Kv satisfies **(ð)**. □

Note that any field with an algebraically closed subfield satisfies **(ð)**. We are now in a position to describe the theories $\mathrm{Th}_\exists(K,v)$ for all $v\in\mathsf{H}^e(K)$.

Theorem 3.5 (Dichotomy). *Let K be a field of characteristic $p\in\mathbb{P}\cup\{0\}$. We have the following dichotomy:*

(i) *Either $\mathsf{H}^e(K)$ is not linearly ordered, in which case $\mathbb{F}_p^{\mathrm{alg}}\subseteq K$, and for all non-trivial $v\in\mathsf{H}^e(K)$ we have that Kv satisfies* **(ð)** *and*
$$\mathrm{Th}_\exists(K,v)=\mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists=\mathbf{H}^{e\prime}(\mathrm{ACF}_p)_\exists.$$
(ii) *Or $\mathsf{H}^e(K)$ is linearly ordered, in which case it admits a smallest element v_K^e, and for all non-trivial $v\in\mathsf{H}^e(K)\setminus\{v_K^e\}$ we have that Kv satisfies* **(ð)** *and*
$$\mathrm{Th}_\exists(K,v)=\mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists=\mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_\exists.$$
Moreover if v_K^e is non-trivial then $\mathrm{Th}_\exists(K,v_K^e)=\mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_\exists$, *and this equals* $\mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ *if and only if Kv_K^e satisfies* **(ð)**.

[4]The letter ð is pronounced "eth".

In particular, Kv does not satisfy **(ð)** *for at most one $v \in \mathsf{H}^e(K)$. If such a v exists, it is the finest element v_K^e of $\mathsf{H}^e(K)$, which is totally ordered.*

Proof. Suppose first that $\mathsf{H}^e(K)$ is not linearly ordered. In this case there exist equicharacteristic henselian valuations v on K with Kv separably closed, for example the canonical henselian valuation v_K. For notational simplicity we denote $\mathbb{F}_0 = \mathbb{Q}$.

Claim 3.5.1. *For every $v \in \mathsf{H}^e(K)$, the residue field Kv contains $\mathbb{F}_p^{\mathrm{alg}}$.*

Proof of claim. Each $v \in \mathsf{H}^e(K)$ is either a refinement of v_K, in which case the residue field Kv is also separably closed and the claim follows, or it is a proper coarsening of v_K. In this latter case, v_K incudes a nontrivial henselian valuation $\bar{v}_K$ on Kv. The residue field Kv_K is separably closed, and so $\mathbb{F}_p^{\mathrm{alg}} \subseteq Kv_K$. By henselianity of $\bar{v}_K$ (see for example [ADF23, Lemma 2.3]), there is a partial section $\zeta : \mathbb{F}_p^{\mathrm{alg}} \to Kv$ of the residue map of $\bar{v}_K$, which proves the claim. $\blacksquare_{\mathrm{claim}}$

Therefore $\mathrm{Th}_\exists(Kv) = \mathbf{ACF}_{p,\exists}$, and so Kv satisfies **(ð)**. Next we suppose that v is nontrivial, so $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ by Lemma 3.4. By Theorem 3.1 we have

$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_\exists = \mathbf{H}^{e\prime}(\mathbf{ACF}_{p,\exists})_\exists = \mathbf{H}^{e\prime}(\mathbf{ACF}_p)_\exists,$$

which finishes case **(i)**.

Suppose that $\mathsf{H}^e(K)$ is linearly ordered. Let v_K^e denote the finest element of $\mathsf{H}^e(K)$: the valuation ring corresponding to v_K^e is the intersection of all the valuations rings of elements of $\mathsf{H}^e(K)$, and such intersections of chains of equicharacteristic henselian valuation rings are again equicharacteristic henselian valuation rings. For every non-trivial $v \in \mathsf{H}^e(K)\setminus\{v_K^e\}$, v_K^e induces a non-trivial equicharacteristic henselian valuation $\bar{v}_K^e$ on Kv. Straight away by Theorem 3.1 we have $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv))_\exists$ and $\mathrm{Th}_\exists(Kv,\bar{v}_K^e) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_\exists$, so that $\mathrm{Th}_\exists(Kv) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{E}}$. Combining these equalities and again applying Theorem 3.1, we have $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{E}})_\exists = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_\exists$. Moreover Kv satisfies **(ð)**, so $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ by Lemma 3.4.

Finally we suppose that v_K^e is non-trivial. By a further application of Theorem 3.1 we have $\mathrm{Th}_\exists(K,v_K^e) = \mathbf{H}^{e\prime}(\mathrm{Th}(Kv_K^e))_\exists$. It only remains to argue that Kv_K^e satisfies **(ð)** if and only if $\mathrm{Th}_\exists(K,v_K^e) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$, but this is simply Lemma 3.4 applied to $v = v_K^e$. □

The **canonical equicharacteristic henselian valuation** v_K^e, which was defined above in case **(i)**, may also be defined in case **(ii)** by writing $v_K^e := v_K$. This allows the rephrasing of Theorem 3.5 as follows:

Theorem 3.6. *Let K be any field. For all non-trivial $v \in \mathsf{H}^e(K) \setminus \{v_K^e\}$, the valued fields (K,v) share the same existential theory. Namely:*

$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists = \mathbf{H}^{e\prime}(\underbrace{\mathbf{H}^{e\prime}(\mathrm{Th}(k))^{\vdash} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}_{\text{the deductive closure in } \mathfrak{L}_{\mathrm{ring}} \text{ of } \mathbf{H}^{e\prime}(\mathrm{Th}(k))})_\exists,$$

where k denotes the residue field of v_K^e.

This theorem has the following corollary.

Corollary 3.7. *For any field K, there are at most three distinct existential theories of the valued fields (K,v), for $v \in \mathsf{H}^e(K)$:*

- **(I)** *the trivial case $v = v_K^{\mathrm{triv}}$,*
- **(II)** *v is non-trivial and Kv satisfies* **(ð)**,
- **(III)** *v is non-trivial and Kv does not satisfy* **(ð)**.

Cases **(I,II,III)** *are always distinct. Case* **(I)** *exists for every K, but both cases* **(II)** *and* **(III)** *may be void, independently, according to the following examples. However, since at most one element of $\mathsf{H}^e(K)$ satisfies* **(III)**, *case* **(II)** *is void if and only if either $\mathsf{H}^e(K) = \{v_K^{\mathrm{triv}}\}$ or $\mathsf{H}^e(K) = \{v_K^{\mathrm{triv}}, v_K^e\}$ with Kv_K^e not satisfying* **(ð)**.

The Main Theorem is also a corollary of these results.

Example 3.8.

- (*a*) Let $K = \mathbb{Q}$. There are no non-trivial elements of $\mathsf{H}^e(\mathbb{Q})$. In this case $\mathbb{Q}$ does not satisfy **(ð)**. Both **(II)** and **(III)** are void.
- (*b*) Let $K = \mathbb{Q}^{\mathrm{alg}}$. There are no non-trivial elements of $\mathsf{H}^e(\mathbb{Q}^{\mathrm{alg}})$. In this case $\mathbb{Q}^{\mathrm{alg}}$ satisfies **(ð)**. Nevertheless, both **(II)** and **(III)** are void.
- (*c*) Let $K = \mathbb{Q}(\!(t)\!)$. The only non-trivial element of $\mathsf{H}^e(\mathbb{Q}(\!(t)\!))$ is v_t, which is in case **(III)** since $\mathbb{Q}$ does not satisfy **(ð)**. Thus **(II)** is void.
- (*d*) Let $K = \mathbb{Q}^{\mathrm{alg}}(\!(t)\!)$. The only non-trivial element of $\mathsf{H}^e(\mathbb{Q}^{\mathrm{alg}}(\!(t)\!))$ is v_t, which is in case **(II)** since $\mathbb{Q}^{\mathrm{alg}}$ satisfies **(ð)**. Thus **(III)** is void.
- (*e*) Let $K = \mathbb{Q}(\!(s)\!)(\!(t)\!)$. There are two non-trivial elements of $\mathsf{H}^e(\mathbb{Q}(\!(s)\!)(\!(t)\!))$, namely v_t and $v_s \circ v_t$. The former, the t-adic valuation v_t has residue field $\mathbb{Q}(\!(s)\!)$, which is henselian, so in particular **(ð)**, thus v_t is in case **(II)**. The latter, the composition $v_s \circ v_t$, has residue field $\mathbb{Q}$ which is in case **(III)**. Note that $\mathbb{Q}$ and $\mathbb{Q}(\!(s)\!)$ have different existential theories, thus so do $(\mathbb{Q}(\!(s)\!)(\!(t)\!), v_t)$ and $(\mathbb{Q}(\!(s)\!)(\!(t)\!), v_s \circ v_t)$.

4. Towards a more general perspective

In [AF26, Section 2], the author, together with Fehm, introduced a framework of contexts, bridges, and arches, in order to systematically study computable interpretations between fragments of theories. Let $A = (B, \hat{B}, \iota)$ be an arch. Suppose ($\star$) that $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$, $\hat{L}_1 = \mathrm{Sent}(\mathfrak{L}_1) \cap \hat{L}_2$, and $L_1 = \mathrm{Sent}(\mathfrak{L}_1) \cap L_2$. We do not suppose ι to be the identity map. The set of L_1-deductively closed L_1-theories forms a complete lattice. Consider the map from $\mathfrak{L}_1$-theories to L_1-theories given by $R \mapsto T_2(R)_{L_1}$, and let Φ_A denote the restriction of this map to L_1-theories.

Definition 4.1. We say that Φ_A is **increasing** if $R \subseteq \Phi_A(R)$, for all L_1-theories R; and that Φ_A is **idempotent** if $\Phi_A(R) = \Phi_A \circ \Phi_A(R)$, for all R.

It is easy to see that $R \subseteq S$ implies $\Phi_A(R) \subseteq \Phi_A(S)$. Now we consider the arch $A = \mathrm{F}_\exists/\mathrm{H}^{e\prime}_\exists \parallel \mathrm{F}/\mathrm{H}^{e\prime}$, using the notation of [AF26], so that Φ_A is now the map

$$\Phi_A(R) = \mathbf{H}^{e\prime}(R)_{\mathrm{E}}.$$

Note that the hypotheses ($\star$) are satisfied for A. We continue to denote $\mathrm{E} = L_1 = \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$ and note that in this case $\iota = \iota_{\mathbf{k}}$ is not the identity map. By Lemma 3.3, a theory $\mathrm{Th}_\exists(k)$ is a fixed point of Φ_A if and only if k satisfies **(ð)**.

It is easy to see that Φ_A is increasing, at least when $R = \mathrm{Th}_\exists(k)$ is the existential theory of a field: since $k \subseteq k(\!(t)\!)$ and $\mathrm{Th}_\exists(k(\!(t)\!)) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(k))_{\mathrm{E}}$, we have $\mathrm{Th}_\exists(k) \subseteq \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(k))_{\mathrm{E}}$.

Lemma 4.2. *Φ_A is increasing.*

Proof. Let R be an E-theory, so that $R = R_\exists$. We want $R \subseteq \mathbf{H}^{e\prime}(R)_{\mathrm{E}}$, i.e. $\mathbf{H}^{e\prime}(R) \models R$. Let $(K, v) \models \mathbf{H}^{e\prime}(R)$. Then $Kv \models R$. By [ADF23] there exists an elementary extension $(K, v) \preceq (K^*, v^*)$ and a partial section $Kv \to K^*$ of the residue map of v^*. Thus $\mathrm{Th}_\exists(Kv) \subseteq \mathrm{Th}_\exists(K^*, v^*)$. Thus $(K^*, v^*) \models R$. □

Lemma 4.3. $\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})} = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$ *for each $\mathfrak{L}_{\mathrm{ring}}$-theory R.*

Proof. Since Φ_A is increasing we have $R \subseteq \mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. Thus $\mathbf{H}^{e\prime}(R) \subseteq \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})$. Let $(K, v) \models \mathbf{H}^{e\prime}(R)$, so that $Kv \models R$. There exists an elementary extension $(K, v) \preceq (K^*, v^*)$ such that v^* admits a nontrivial proper coarsening w. Then $K^*v^* \models R$, so $(K^*w, \bar{v}) \models \mathbf{H}^{e\prime}(R)$ and $K^*w \models \mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. Therefore $(K^*, w) \models \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})$, and so $K^* \models \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. This proves that $\mathbf{H}^{e\prime}(R) \models \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. □

Lemma 4.4. *Φ_A is idempotent.*

Proof. Let R be an E-theory. Taking the existential consequences of the previous lemma we have $\mathbf{H}^{e\prime}(R)_{\mathrm{E}} = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{E}}$. The latter is equal to $\mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(R)_{\mathrm{E}})_{\mathrm{E}}$. □

Observation 4.5. *Let R be an* E*-theory of fields. The following are equivalent.*

(a) *R is a fixed point of Φ_A.*
(b) *R is in the image of Φ_A.*
(c) *Every model of R is $\mathbb{Z}$-large and $R = R_\exists$.*

Proof. The equivalence **(a)**⇔**(b)** is trivial since Φ_A is idempotent. For **(c)**⇒**(a)**: by monotonicity we have $R \subseteq \mathbf{H}^{e\prime}(R)_{\mathrm{E}}$. If $F \models R$ then F is $\mathbb{Z}$-large, so $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_{\mathrm{E}}$ by Lemma 3.3. Therefore $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_{\mathrm{E}} \supseteq \mathbf{H}^{e\prime}(R)_{\mathrm{E}}$, and so $R \models \mathbf{H}^{e\prime}(R)_{\mathrm{E}}$. Since also $R = R_{\mathrm{E}}$, we have $R_{\mathrm{E}} = \mathbf{H}^{e\prime}(R)_{\mathrm{E}}$, which proves **(a)**. Next, suppose **(a)** and let $F \models R = \mathbf{H}^{e\prime}(R)_{\mathrm{E}}$. Then F is a model of the existential $\mathfrak{L}_{\mathrm{ring}}$-theory of a henselian field, thus it is $\mathbb{Z}$-large. □

Question 4.6. What more can we say about $\mathfrak{L}_{\mathrm{ring}}$-theories R that are fixed points of Φ_A?

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