Dimensions in model-theoretic algebra

Adrien Deloro

Sorbonne Université

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Stand up for Science!

A map of logic?

Classically:

- proof theory
- recursion theory, incl. complexity theory+reverse math.
- set theory
- model theory ($\leftarrow$ you are here)

Add at least:

- algebraic logic
- categorical logic
- non-classical logics
- human/machine interface, theorem checking/proving
- applications to linguistics, philosophy, ... mathematics

A map of model theory?

- basic model theory (useful for all branches of logic)
- 'abstract model theory' (AEC's, abstract logics...)
- connections to other branches of logic (recursion: Scott theory, complexity: finite model theory)
- classification theory (Shelah), neostability
- applied model theory, incl. model-theoretic algebra ($\leftarrow$ you are here)

This is a talk in *model-theoretic algebra*.

Model-theoretic algebra?

Model theory $\leftrightsquigarrow$ algebra
Or: model theory, as a toolbox for 'tame' algebraic structures.
Goes back a long way: Malcev (1930s!), Robinson, Tarski...
Typical statements in model-theoretic algebra take two forms.

- Logical classification results:
  'let $\mathbb{A} = \mathrm{SL}_2(\mathbb{C})$. Then $\mathrm{Th}(\mathbb{A})$ is $\aleph_1$-categorical.'
- Algebraic classification results:
  'let $\mathbb{A}$ be an infinite field definable in an *o*-minimal theory. Then $\mathbb{A}$ is alg. closed or real closed.'

We focus on 'algebraic classification results'.

Structures of interest in model-theoretic algebra

Here is the kind of things we look at:

- fields
- associative rings
- *Lie rings*
- groups

Since this is one of my current interests, here is the definition.

Definition

A *Lie ring* (= 'a Lie algebra without a base field') is an abelian group $(\mathfrak{g}; +)$ equipped with bi-additive $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

- $[x, x] = 0$;
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi).

I want to survey a (too?) general notion and two special cases of interest.
Key idea: *strong model-theoretic assumptions* $\Rightarrow$ *known objects.*

I. Dimensional theories

A recent, very general framework for model-theoretic algebra

Definable and interpretable sets

Definition

Let $\mathbb{M}$ be a structure.

- A *definable* set is some $X \subseteq \mathbb{M}^k$ such that there is $\varphi(\underline{x}, \underline{a})$ with

$$X = \varphi[\mathbb{M}] = \{\underline{m} \in \mathbb{M}^k : \mathbb{M} \models \varphi(\underline{m}, \underline{a})\}.$$

- An *interpretable set* is some X/R where $X \subseteq \mathbb{M}^k$ is definable, and $R \subseteq X^2$ is (graph of) definable eq. relation.

- Parameters (tuple $\underline{a}$) implicitly allowed.
- There is a trick ('Shelah's eq') to conflate interpretable with definable. (vanilla: do not worry)
- These 'sets' are actually functors from models of $\mathrm{Th}(\mathbb{M})$ to sets.

Dimensional theories

Let T be a theory. Let $\mathrm{Def}(T) = \{$interpretable sets$\}$.

Definition (Wagner 2020)

T is *dimensional* if there is $\dim\colon \mathrm{Def}(T) \setminus \{\emptyset\} \to \mathbb{N}$ with:

(X, Y stand for definable sets)

- $\dim X = 0$ iff X is finite;
- if $f\colon X \simeq Y$ is interpretable bijection, then $\dim X = \dim Y$;
- $\dim(X \cup Y) = \max(\dim X, \dim Y)$;
- if $f\colon X \to Y$ is interpretable and $(\forall y \in Y)(\dim f^{-1}(\{y\}) = d)$, then $\dim X = d + \dim Y$.

First attempted definition by van den Dries 1989.
Really took off with Wagner 2020 (which has many variations).

Examples

Axioms for dimension:

- $\dim X = 0$ iff X is finite;
- if $f\colon X \simeq Y$ is bijection, then $\dim X = \dim Y$;
- $\dim(X \cup Y) = \max(\dim X, \dim Y)$;
- if $f\colon X \to Y$ has $(\forall y \in Y)(\dim f^{-1}(\{y\}) = d)$, then $\dim X = d + \dim Y$.

Examples (fully described later):

- Let T be *o*-minimal theory (=theory of an *o*-min. structure). Then T is dimensional (van den Dries, Pillay, Steinhorn).
- Let T be theory of an $\aleph_1$-categorical alg. structure. Then T is dimensional (Morley, Baldwin).

(Warning: these are non-generic examples as they encode more information than a dimension, a *combinatorial geometry*/*matroid*.)

Model-theoretic dimensional algebra

Key idea: *strong model-theoretic assumptions* $\Rightarrow$ *known objects*

Little is known. Let T be dimensional.

- Let $\mathbb{F}$ be an infinite skew-field definable in T.
  Then $\mathbb{F}$ is finite over $Z(\mathbb{F})$ ($= \mathbb{F}$ close to being commutative).
- Let G be a group of dimension ≤ 3. Assume G pseudofinite.
  Then G virt. soluble or virt. isom. $\mathrm{PSL}_2(\mathbb{F})$ (Karhumäki-Wagner).
- Simple Lie rings: under study (ongoing, Invitti).

Clearly, more could be attempted. However...

Skepticism

Dimensional algebra begs to be pushed, but expect no miracles.

- Assumptions are weak.
  Categorical theories and *o*-minimal theories encode a combinatorial geometry, while abstract 'dimensional' setting does not.
- Historical precedent: stability, once thought a paradise for model-theoretic algebra.
  Proved wrong by: 1. Sela's stability of the free group, and 2. *o*-minimality (unstable, yet works like a charm).

To me, a crash test could be: Sylow 2-theory/no Sylow 2-theory.

II. *o*-minimal theories

A context tailored for ‘tame real’ geometry

o-minimal theories are dimensional

Definition (van den Dries, Pillay-Steinhorn 1980s)

An ordered structure $\mathbb{M}$ is *o-minimal* if every definable $X \subseteq \mathbb{M}^1$ is a finite union of intervals.

Example

- $(\mathbb{R}; <, +, \cdot)$ (Tarski)
- $(\mathbb{R}; <, +, \cdot, \{\text{restricted analytic functions}\})$ (Gabrielov)
- $(\mathbb{R}; <, +, \cdot, \exp)$ (Wilkie)

Key facts:

- if $\mathbb{M}$ is *o*-minimal, then any $\mathbb{M}' \equiv \mathbb{M}$ is. Say '$\mathrm{Th}(\mathbb{M})$ is *o*-minimal'.
- If T is *o*-minimal, then any $X \in \mathrm{Def}(T)$ decomposes into *cells*.
- T is dimensional.

Warning: T even encodes a 'pregeometry' (matroid).

The success of *o*-minimality

o-minimality was not created by nor for algebraists.

- van den Dries saw it as model theory's response to Grothendieck's call for a 'tame topology';
- Peterzil-Starchenko proved (*o*-min. analogue of) Zilber's trichotomy: *pregeometries in o-minimal structures are either trivial, or vector-space-like, or field-like*.
- *o*-min. applies to number theory ('Pila-Wilkie counting theorem').

However, I'll focus on model-theoretic *o*-minimal algebra.

Model-theoretic *o*-minimal algebra

Key idea: *strong model-theoretic assumptions* $\Rightarrow$ *known objects*

A lot is known. Let T be *o*-minimal.

- Let $\mathbb{F}$ be an infinite skew-field definable in T.
  Then $\mathbb{F}$ is real closed, alg. closed, or quaternion-like (Pillay-Steinhorn).
- Let R be an infinite, simple associative ring definable in T.
  Then R is $M_n(\mathbb{F})$ for some def. field $\mathbb{F}$ (folklore).
- Let $\mathfrak{g}$ be an infinite, simple Lie ring def. in T.
  Then $\mathfrak{g}$ is a Lie algebra over some def. field (folklore).
- Let G be an infinite, simple group def. in T.
  Then G is Lie-like over some def. field (Pillay-Peterzil-Starchenko).

o-minimal world enables some classical tools.

Let G be a simple group def. in an *o*-minimal theory.

1. Cell decomposition gives a *definable Euler characteristic*,
2. which yields good control over torsion in abelian groups,
3. implying linearisation results:
   if G acts on abelian A, then A is a vector space over a field.
4. Also one has definable-differential methods:
   if G is definable group, then G acts on tangent space $T_1G\dots$
5. so one retrieves a Lie ring $\mathfrak{g}$.
6. By (3) $\mathfrak{g}$ is a Lie algebra,
7. and at this stage, not hard to learn something about G.

Dimensional: not enough. | *o*-minimal: overkill. | We go to middle grounds.

III. Ranked structures

At the intersection of model theory, algebraic geometry. . . and pure algebra

The origin

- Topic starts with Morley's theorem:
  a theory T is κ-categorical for some $\kappa \geq \aleph_1$ iff it is for all $\kappa \geq \aleph_1$.
- *Added after talk: Marsh 1966 is first 'geometric' investigation into strong minimality.* (Thanks to M. D. for pointing that out.)
- Pushed further by Baldwin:
  if T is $\aleph_1$-categorical, then Morley's 'rank' takes finite values.
- 70s folklore: *in alg. closed fields, Morley rank=Zariski dimension.*
- So one could conjecture: $\aleph_1$*-categoricity=algebraic geometry* (false).
- In particular: *simple $\aleph_1$-categorical groups=simple alg. groups* (open).

However, I'll offer something anachronistic (and algebra-oriented).

Borovik's take at Morley rank

Borovik (a group theorist) suggested the following.

Definition

A structure $\mathbb{M}$ is *ranked* if there is $\mathrm{Def}(\mathbb{M}) \setminus \{\emptyset\} \to \mathbb{N}$ satisfying:

- $\dim X \geq n+1$ iff there are disjoint $Y_1, Y_2, \ldots, \subseteq X$, all of $\dim \geq n$;
- for $f\colon X \to Y$, each $Y_k = \{y \in Y : \dim f^{-1}(\{y\}) = k\}$ is in $\mathrm{Def}(\mathbb{M})$;
- if $Y = Y_k$ above, then $\dim X = \dim Y + k$;
- for $f\colon X \to Y$, there is k s.t.:
  $(\mathrm{card}\, f^{-1}(\{y\}) \geq k) \to (\mathrm{card}\, f^{-1}(\{y\}) \geq \aleph_0)$.

Messieurs Borovik, Morley, Poizat (for connoisseurs)

Borovik-Poizat axioms:

1. $\dim X \geq n+1$ iff there are disjoint $Y_1, Y_2, \ldots, \subseteq X$, all of $\dim \geq n$;
2. for $f\colon X \to Y$, each $Y_k = \{y \in Y : \dim f^{-1}(\{y\}) = k\}$ is in $\mathrm{Def}(\mathbb{M})$;
3. if $Y = Y_k$ above, then $\dim X = \dim Y + k$;
4. for $f\colon X \to Y$, there is k s.t. $(\mathrm{card}\, f^{-1}(\{y\}) \geq k) \to (\mathrm{card}\, f^{-1}(\{y\}) \geq \aleph_0)$.

Experts: (1) *looks* like the definition of Morley rank but differs.

- Morley rank is about definable sets, not all interpretable sets.
- Borovik is about *structure*, not about theory (no saturation!)

However Poizat proved that for a group, Borovik=Morley.
Striking connections for groups:

- G is ranked iff $\mathrm{Th}(G)$ has finite Morley rank (Poizat)
- Suppose G simple. Then $\mathrm{Th}(G)$ has finite Morley rank iff $\mathrm{Th}(G)$ is $\aleph_1$-categorical (Zilber, a converse to Baldwin!)

Model-theoretic ranked algebra

Key idea: *strong model-theoretic assumptions* $\Rightarrow$ *known objects*

Things get really interesting (to me). Let T be a ranked theory.

(Because I work with groups+, 'ranked'='finite Morley rank' by Poizat.)

- Let $\mathbb{F}$ be an infinite skew-field def. in T.
  Then $\mathbb{F}$ is commutative alg. closed field (Macintyre).
- Let R be a simple, associative ring def. in T.
  If R has characteristic 0, then R is $M_n(\mathbb{F})$ for some def. $\mathbb{F}$ (Zilber).
- Let $\mathfrak{g}$ be a simple, torsion-free Lie ring def. in T.
  Then $\mathfrak{g}$ is Lie algebra over some def. $\mathbb{F}$ (Zilber).
- Let G be an infinite simple group def. in T.
  Cherlin-Zilber conjecture: G *should be* alg. group over some def. $\mathbb{F}$ (open).

Cherlin-Zilber conjecture

Conjecture (Cherlin-Zilber)

Let G be an infinite, simple, ranked group. Then G is an alg. group over some def. $\mathbb{F}$.

Challenging:

- no topology a priori
- no infinitesimals a priori
- no field a priori (Zilber's trichotomy being false)

Naive attempt: induction. Known in rank ≤ 3 (meaning, open in rank 4).
Borovik: let's imitate *classification of finite simple groups*.
Still challenging:

- no counting arguments, character theory, etc.

To-date, still our best idea. (Partial, positive answers in arbitrary rank.)

A current development

Conjecture (Cherlin-Zilber)

Let G be an infinite, simple, ranked group. Then G is an alg. group over some def. $\mathbb{F}$.

Conjecture

Let $\mathfrak{g}$ be an infinite, simple, ranked Lie algebra. Then G is a Lie algebra over some def. $\mathbb{F}$.

Known in rank ≤ 4 (D.-Tindzogho Ntsiri, 2024; notice $4 > 3$).
However, I do not conjecture any relation between both conjectures (one would need more model theory).

A conclusion?

- Model theory is a language for 'tame algebra';
- model-theoretic algebra *uses* model-theoretic notions, and does algebra;
- many challenging questions there.

'**M**odel-theoretic **A**lgebra and **G**eometric **A**lgebra!'

Thank you