Dimensions in model theory

Adrien Deloro

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Abstract. • This is a survey talk in model theory, aimed at a general audience. • One needs to know what first-order logic is, and the notion of a model. No further knowledge of model theory is required. • I will explain how two quite different branches of model theory naturally merge into the same topic: theories with a dimension function. This paves the way to a second talk.

Outline

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1 Categoricity

1.1 Categoricity lost

One of the origins of modern logic is the renewal of the axiomatic method in the second half of the XIX$^{\text{th}}$ century.

Theorem (Dedekind 1888 [Ded88])**.** *Second-order arithmetic has a unique model up to isomorphism.*

Recall that second order-order arithmetic is the second-order theory in the language $\{0, s\}$ expressing:

- s is a unary, injective function and every $x \neq 0$ is an image;
- every SUBSET of the domain containing 0 and closed under s is the whole domain.

Definition (Veblen 1904 [Veb04]; name suggested by Dewey)**.** *A theory is* (absolutely) categorical *if it has a unique model up to isomorphism.*

Example 1. The first-order theory of any finite structure. Say $\mathbb{M} = \underline{m}$ as a tuple. Consider:

$$(\exists \underline{x}) \left[\left(\bigwedge_{\substack{\text{atomic form.}\\ \text{true of } \underline{m}}} \varphi(\underline{x}) \right) \wedge \left(\bigwedge_{\substack{\text{atomic form.}\\ \text{false of } \underline{m}}} \neg\varphi(\underline{x}) \right) \wedge (\forall y) \left(\bigvee_{i=1}^{n} y = x_i \right) \right].$$

This is not very interesting (except in finite model theory).

[Ded88] : Richard Dedekind. *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg und Sohn, 1888. xv+58

[Veb04] : Oswald Veblen. 'A system of axioms for geometry'. *Trans. Amer. Math. Soc.* 5.3 (1904), pp. 343–384

Example 2. Second-order arithmetic (Dedekind).

Example 3 (Huntigton 1903 [Hun03])**.** The second-order theory of $(\mathbb{R}; <, +, \cdot)$. It is given by:

- ordered field;
- every non-empty, bounded above SUBSET of the domain has a least upper bound.

Both Dedekind and Huntigton were using 'full semantics', viz. with the 'full' notion of a SUBSET. (This contrasts with 'Henkin semantics', if you know what that is.) However, as intuitioned by Weyl and confirmed by XX$^{\text{th}}$ work on set theory, 'full semantics' is not intrinsic to mathematical structures but heavily depends on how they are embedded into models of set theory. Thus, 'full semantics' is by no means a reliable tool—and certainly not one suitable for the foundational purposes the axiomatic school had in mind.

Arguably the birth of mathematical logic is the discovery of relativism.

Theorem (Löwenheim 1915 [Löw15], Skolem 1920 [Sko20] and 1922 [Sko23], Tarski 1934 as a note to [Sko34], Malcev 1936 [Mal36])**.** *Let T be a countable first-order theory having an infinite model. Then for every cardinal $\kappa \geq \aleph_0$, T has a model of cardinal κ.*

This created havoc in foundational studies: if consistent, ZF has a countable model ('Skolem's paradox'). To us, it merely demonstrates the limited strength of first-order logic. As a conclusion:

- 'full semantics' has no intrinsic meaning, and first-order logic must be prefered;
- in first-order logic, only theories of finite structures (example 1 above) can be absolutely categorical.

1.2 Categoricity regained

Returning to Löwenheim-Skolem phenomena, it looks like cardinality is the main obstacle to absolute categoricity.

Definition (Łoś 1952 [Łoś54])**.** *Let T be a theory and $\kappa \geq \aleph_0$ be a cardinal. T is* κ-categorical *if it has a unique model of cardinal κ up to isomorphism.*

From now on we focus on complete theories. The reason to do so is the following.

Theorem (Łoś; Vaught 1954 [Vau54])**.** *Let T be a first-order theory with no finite model. If there is $\kappa \geq \aleph_0$ such that T is κ-categorical, then T is complete.*

[Hun03] : Edward Huntington. 'Complete sets of postulates for the theory of real quantities'. *Trans. Amer. Math. Soc.* 4.3 (1903), pp. 358–370

[Löw15] : Leopold Löwenheim. 'Über Möglichkeiten im Relativkalkül'. *Math. Ann.* 76.4 (1915), pp. 447–470

[Sko20] : Thoralf Skolem. *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen.* Kristiania, 1920. 36 pp.

[Sko23] : Thoralf Skolem. 'Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre'. 5. Kongreß Skandinav. in Helsingfors vom 4. bis 7. Juli 1922. Helsinki: Akademiska Bokhandeln, 1923, pp. 217–232

[Sko34] : Thoralf Skolem. 'Über die Nichtcharakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschließlich Zahlenvariablen'. *Fundam. Math.* 23 (1934), pp. 150–161

[Mal36] : Anatoli Maltsev. 'Untersuchungen aus dem Gebiet der mathematischen Logik'. *Rec. Math. Moscou, n. Ser.* 1 (1936), pp. 323–336

[Łoś54] : Jerzy Łoś. 'On the categoricity in power of elementary deductive systems and some related problems'. *Colloq. Math.* 3 (1954), pp. 58–62

[Vau54] : Robert Vaught. 'Applications of the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability'. *Nederl. Akad. Wetensch. Proc. Ser. A.* **57** = *Indagationes Math.* 16 (1954), pp. 467–472

(What happens in second-order 'full semantics' is quite unsettling and worth a dedicated study.) So this leaves out theories like PA or ZF. We give examples of 'categoricity in power'.

Example 1 (infinite sets)**.** The theory of infinite, amorphous sets in $\{=\}$:

$$T_\infty = \text{for each } n, \text{ axiom } (\exists x_1)\dots(\exists x_n)\left(\bigwedge_{i\neq j} x_i \neq x_j\right).$$

By definition of a cardinal, T_∞ is κ-categorical for each $\kappa \geq \aleph_0$.

Example 2 (dense linear orderings)**.**

$$\mathrm{DLO} = \begin{cases} < \text{ is a linear ordering,} \\ \text{no least nor greatest element,} \\ (\forall x)(\forall y)[x < y \to (\exists z)(x < z < y)] \end{cases}$$

Then DLO is $\aleph_0$-categorical (Cantor 1895 [Can95]). Is is not κ-categorical for any $\kappa > \aleph_0$ (possibly Hausdorff).

Example 3 (vector spaces)**.** Let $\mathbb{F}$ be a fixed, countable field. Language: $\{+\} \cup \{\lambda_a : a \in \mathbb{F}\}$.

$$\mathbb{F}\text{-Vect} = \begin{cases} \text{abelian group} \\ \text{for each } a \in \mathbb{F}, \text{ axioms } (\forall x)(\forall y)(\lambda_a(x+y) = \lambda_a(x) + \lambda_a(y)), \text{etc.} \end{cases}$$

Not $\aleph_0$-categorical since $\mathbb{F} \not\simeq \mathbb{F}^2$. But κ-categorical for each $\kappa > \aleph_0$; this is essentially *dimension theory*.

Let me give a more algebraic twist to the same example.

Example 3' (algebraically closed fields of a given characteristic)**.**

$$\mathrm{ACF} = \begin{cases} \text{field,} \\ \text{for each } n \geq 1, \text{ axiom } (\forall a_{n-1})\dots(\forall a_0)(x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0) \end{cases}$$

For p a prime,

$$\mathrm{ACF}_p = \mathrm{ACF} \cup \{\underbrace{1 + \dots + 1}_{p} = 0\};$$

$$\mathrm{ACF}_0 = \mathrm{ACF} \cup \{\underbrace{1 + \dots + 1}_{p} \neq 0 : p \text{ a prime}\}.$$

For q a prime or 0, ACF_q is not $\aleph_0$-categorical ($\overline{\mathbb{F}_q(X)} \not\simeq \overline{\mathbb{F}_q}$). For each $\kappa \geq \aleph_1$, it is κ-categorical (Steinitz 1910 [Ste10]). Again, this is essentially dimension theory.

It is easy to produce complete theories not κ-categorical in any κ (and RCF of section 2 is one such example). However, behaviour at $\aleph_1$, $\aleph_2$, etc. seems to be related. Łoś conjectured that first-order logic cannot distinguish 'higher infinites', viz. that all cardinals $\geq \aleph_1$ have the same categoricity behaviour.

[Can95] : Georg Cantor. 'Beiträge zur Begründung der transfiniten Mengenlehre I'. *Math. Ann.* 46 (1895), pp. 481–512

[Ste10] : Ernst Steinitz. 'Algebraische Theorie der Körper'. *J. Reine Angew. Math.* 137 (1910), pp. 167–309

1.3 Categoricity and dimension

Theorem (Morley 1965 [Mor65])**.** *Let T be a countable first-order theory. Then (there is $\kappa \geq \aleph_1$ such that T is κ-categorical) iff (for all $\kappa \geq \aleph_1$, T is κ-categorical).*

(So we just say '$\aleph_1$-categorical'.) Countability assumption later removed by Shelah. This is of course splendid, but even more is the proof. Morley introduced a form of *abstract dimension* on definable sets of T. This was quickly isolated.

Theorem (Baldwin 1973 [Bal73])**.** *Let T be a countable, $\aleph_1$-categorical theory. Then definable sets of T bear an integer-valued dimension.*

This dimension even relates to a geometry (in the sense of combinatorial geometry).

Conjecture (Zilber before 1984; false)**.** *Let T be $\aleph_1$-categorical. Suppose further that every definable set in one variable is either finite or cofinite. Then:*

- *either T has essentially no 'geometry';*
- *or T has essentially the same geometry as a $\mathbb{F}$-vector space;*
- *or T has essentially the same geometry as an* ACF*.*

This has been proved wrong by Hrushovski 1993 [Hru93], but correct under extra assumptions (Hrushovski-Zilber 1993 [HZ93]). As a conclusion:

- $\aleph_1$-categoricity seemingly relates to 'tame geometry';
- it implies dimensionality.

And now for something completely different.

2 o-minimality

2.1 Euclid completed

This part of the story starts in Greece of course, continues with Descartes 1637 [Des37], and resumes with the renewal of the axiomatic method again. Hilbert 1899 [Hil99] produced a second-order, infinitary axiomatisation of the Euclidean plane. In 1900, he also proposed a second-order axiomatisation of the real ordered field. The first-order study of the reals goes back to the 1920s, with Tarski of course, but also of Artin and Schreier.

Example 1 (real closed fields)**.**

$$\mathrm{RCF} = \begin{cases} \text{field with compatible ordering,} \\ \text{every positive element a square,} \\ \text{for each odd } n\text{, axiom 'polynomials of degree } n \text{ have roots'} \end{cases}$$

Not κ-categorical for any $\kappa \geq \aleph_0$ (play with infinitesimals and omitting types).

[Mor65] : Michael Morley. 'Categoricity in power'. *Trans. Amer. Math. Soc.* 114 (1965), pp. 514–538
[Bal73] : John Baldwin. 'α_T is finite for $\aleph_1$-categorical T'. *Trans. Amer. Math. Soc.* 181 (1973), pp. 37–51
[Hru93] : Ehud Hrushovski. 'A new strongly minimal set'. Vol. 62. 2. Stability in model theory, III (Trento, 1991). 1993, pp. 147–166
[HZ93] : Ehud Hrushovski and Boris Zilber. 'Zariski geometries'. *Bull. Amer. Math. Soc. (N.S.)* 28.2 (1993), pp. 315–323
[Des37] : René Descartes. *Discours de la méthode*. Leyde: Jan Maire, 1637. 413+xxxii
[Hil99] : David Hilbert. *Grundlagen der Geometrie*. Leipzig: Teubner, 1899. 92 pp.

Theorem (Tarski 1926-1927?)**.**

(i) There is a natural, effective, complete theory axiomatising ($\mathbb{R}$;*between-ness, equality of distances*)*.*

(ii) This theory is bi-interpretable with RCF*.*

(Actually (ii) is the origin of Tarski's first interest in RCF, and heavily uses Hilbert's results.) So a second look at RCF is in order. Notice that since RCF enjoys no κ-categoricity, this is not a repeat of section 1.

2.2 o-minimal structures

Theorem (Tarski 1948, with origins in the 1930s)**.**

(i) RCF *eliminates quantifiers, viz. for any formula* $\varphi(\underline{x})$ *there is quantifier-free* $\varphi_0(\underline{x})$ *with* $\mathrm{RCF} \models (\forall \underline{x})(\varphi(\underline{x}) \leftrightarrow \varphi_0(\underline{x}))$*.*

(ii) In particular, if $\mathbb{M} \models \mathrm{RCF}$ *and* $X \subseteq \mathbb{M}$ *is definable, then* X *is a finite union of intervals.*

(i) is not obvious but (ii) is fairly straightforward from it. A singleton $\{a\}$ is the interval $[a, a]$.

The topic remained relatively confidential in model theory (also because of the brilliance of Morley and then Shelah), until a quick boom in the 1980s.

Definition (van den Dries 1984 [Dri84] ; Pillay-Steinhorn 1986 [PS86])**.** *Let* $(\mathbb{M}; <, \cdot)$ *be an ordered structure.* $\mathbb{M}$ *is* o-minimal *if every definable subset* $X \subseteq \mathbb{M}$ *is a finite union of intervals.*

This directly aims at capturing 'real, tame' geometry.

Example 1 (Tarski)**.** $(\mathbb{R}; <, +, \cdot)$ is o-minimal.

Example 2 (Gabrielov 1968 [Gab68])**.** Add a function symbol for each restriction $f_{|[0,1]}$, where f is analytic. Then $(\mathbb{R}; <, +, \cdot, \{\text{all restricted analytic functions}\})$ is o-minimal.

Example 3 (Wilkie 1996 [Wil96])**.** $(\mathbb{R}; <, +, \cdot, \exp)$ is o-minimal (with *unrestricted* exp).

Since, o-minimality has had striking applications and is one of the meeting points of model theory and number theory.

2.3 o-minimality and dimension

Notice how o-minimality was defined for a structure; at this point it hardly qualifies as mathematical logic.

Theorem (Pillay-Steinhorn)**.** *Let* $(\mathbb{M}; <, \dots)$ *be o-minimal. Then:*

(i) any $\mathbb{M}' \equiv \mathbb{M}$ *is o-minimal, so this really is a property of* $\mathrm{Th}(\mathbb{M})$*;*

[Dri84] : Laurentius van den Dries. 'Remarks on Tarski's problem concerning $(\mathbb{R}; , +, \cdot, \exp)$'. *Logic colloquium '82 (Florence, 1982)*. Vol. 112. Stud. Logic Found. Math. North-Holland, Amsterdam, 1984, pp. 97–121

[PS86] : Anand Pillay and Charles Steinhorn. 'Definable sets in ordered structures I'. *Trans. Amer. Math. Soc.* 295.2 (1986), pp. 565–592

[Gab68] : Andrei Gabrielov. 'Projections of semianalytic sets'. *Funkcional. Anal. i Priložen.* 2.4 (1968), pp. 18–30

[Wil96] : Alex Wilkie. 'Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function'. *J. Amer. Math. Soc.* 9.4 (1996), pp. 1051–1094

(ii) every definable $X \subseteq \mathbb{M}^k$ can be written as a finite union of simple definable sets called 'cells';

(iii) in particular, definable sets bear an integer-valued dimension.

Here again, the dimension relates to a combinatorial geometry. Although *o*-minimal dimension and Morley rank do not coincide, they share many common 'geometric' features.

As a conclusion:

- *o*-minimality was designed to capture 'real, tame geometry';
- it implies dimensionality.

3 Coda: Dimensional theories

3.1 Dimensions from pregeometries

Definition. *A* pregeometry (X, cl) *is a set X with a closure operator* $\mathrm{cl}\colon P(X) \to P(X)$*, such that for all $A \in P(X)$ and $a, b \in X$ one has:*

- $A \subseteq \mathrm{cl}(A)$ *(reflexivity);*
- $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ *(transitivity);*
- $\mathrm{cl}(A) = \bigcup_{finite\ F \subseteq A} \mathrm{cl}(F)$ *(locality);*
- *if $a \in \mathrm{cl}(A, b) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(A, a)$ (Steinitz' exchange principle).*

It so happens that both *o*-minimal and strongly minimal theories encode such a pregeometry. Using the exchange principle, a pregeometry gives rise to a dimension function. This finally explains why both $\aleph_1$-categorical nature and *o*-minimal nature bear an integer-valued dimension function on definable sets.

3.2 Dimensions in the wild

Definition. *Let $\mathbb{M}$ be a structure.*

- *A* definable set *is a subset $X \subseteq \mathbb{M}^k$ of the form $X = \{\underline{m} \in \mathbb{M}^k : \mathbb{M} \models \varphi(\underline{m}, \underline{a})\}$ for some first-order formula (with parameters) $\varphi(\underline{x}, \underline{a})$.*
- *An* interpretable set *is a quotient X/R where $X \subseteq \mathbb{M}^k$ is definable, and $R \subseteq X^2$ is a definable equivalence relation.*

Here we allowed quotients because the next talk will be more algebraic. The removal of quotients is called 'elimination of imaginaries' (a technical topic in model theory).

The following should be called 'dimensions without a geometry'.

Definition. *A theory T is* dimensional *if there is a integer-valued dimension function on the definable sets of its models, satisfying natural axioms.*

... to be continued