Axiomatisabilité et décidabilité de corps valués henséliens

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Abstract

In this mémoire we explore model theoretic questions around the axiomatizability and decidability of theories and fragments of theories of henselian valued fields, particularly those of positive residue characteristic.

Perhaps the most important open problem in this area is the question of the decidability of fields $\mathsf{F}_q((t))$ of formal power series over finite fields. While the full theories of such fields remain mysterious, their existential theories are increasingly well understood: in 2003, Denef and Schoutens gave an algorithm, based on the assumption of Resolution of Singularities in positive characteristic, to determine whether or not an existential sentence in the language of rings together with a symbol for t holds in $\mathsf{F}_q((t))$.

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With Fehm, in 2016, we gave unconditionally a decision procedure to decide the existential theory of any henselian valued fields in equal characteristic, in a language *without* additional symbols, e.g. for t. Later in 2023, additionally with Dittmann, we gave a decision procedure to decide the existential theory with t conditionally on an assumption called **(R4)**, which follows from Resolution of Singularities, but seems in principle significantly weaker. In the mémoire, we present a more unified exposition of these two results.

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Two other topics briefly explored in this talk are "existentially t-henselian" fields—those fields existentially equivalent to a field admitting a nontrivial henselian valuation—, and theories of separably tame valued fields, which we extend to allow infinite imperfection degree.

Today

1. Setting the scene: axiomatizability and decision problems
2. Preliminaries
- Valued fields
- Henselianity
3. The existential theory of $F((t))$
- Notation
4. Uniformities (MThms 1, 2, & 3)
- A different perspective
- What is the algebraic input?
- Consequences for decidability
- $\mathbf{H}^{e0}$! $\mathbf{F}$ and 9_n ! 9_n
5. Towards universal-existential
- $\mathbf{H}^{e;Z}$! $\mathbf{F}$ and $8_1 9$! $8_1 9$
- A failure of monotonicity
- Summary
6. Separably tame valued fields (MThm 6)
7. Mixed characteristic, finitely ramified (MThm 4)
8. The canonical eq-char henselian val & Z-largeness (MThm 5)

Setting the scene: axiomatizability and decision problems

Straightforward questions?

Which one-variable polynomial equations over $\mathbb{Z}$ have solutions

1. in $\mathbb{C}$?
2. in $\mathbb{F}_p$?
3. in $\mathbb{R}$?
4. in $\mathbb{Q}$?
5. in $\mathbb{Z}$?
6. in $\mathbb{N}$?

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Which one-variable polynomial equations over $\mathbb{Z}$ have solutions

1. in $\mathbb{C}$? whenever degree > 0
2. in $\mathbb{F}_p$? whenever degree > 0 mod p
3. in $\mathbb{R}$? e.g. roots are computable + induction
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We could allow many variables, and perhaps "arbitrary" coefficients.
E.g. what about roots in $\mathbb{C}$ of multi-variable polynomials with complex coefficients?

Setting the scene: axiomatizability and decision problems

Straightforward questions?

Which one-variable polynomial equations over Z have solutions

1. in C?	whenever degree $>$ 0
2. in $\bar{F}_p$?	whenever degree $>$ 0 mod p
3. in R?	e.g. roots are computable + induction
4. in Q?	Gauss's Lemma + direct search
5. in Z?	similarly
6. in N?	similarly

We could allow many variables, and perhaps "arbitrary" coefficients.
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Hilbert's Tenth Problem (**H10**) for R (original version is $R = Z$)

Give an algorithm (=Turing machine) to decide correctly, for each $f\ 2\ Z[X_1; : : : ; X_n]$, whether the Diophantine equation

$$f(X_1; :::; X_n) = 0$$

has a solution in R.

Stronger versions require the algorithm to handle all $f\ 2\ S[X_1; : : : ; X_n]$ for various $S \quad R$.

Setting the scene: axiomatizability and decision problems

Theorem (Davis–Putnam–Robinson–Matiyasevich, 1949-70)

H10 *for* $\mathbb{Z}$ *is unsolvable, i.e. there is no such algorithm.*

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H10 *for Z is unsolvable, i.e. there is no such algorithm.*

In this talk, a formula is **existential** if it is of the form

$$\exists x_1 \ldots \exists x_m \quad (x_1, \ldots, x_m, y_1, \ldots, y_n),$$

for a quantifier-free formula .

So the DPRM theorem says that $\mathrm{Th}_\exists(Z, +, \ , \ , 0, 1)$ is undecidable.

For most of the rings/fields we would ever consider,

- **H10** for R is equivalent to the decidability of the existential theory $\mathrm{Th}_\exists(R)$ of R.
- the stronger versions of **H10** for R are equivalent to the decidability of the existential theory of R in a language augmented certain extra constant symbols for $s \in S$.

Setting the scene: axiomatizability and decision problems

Classical results

1. Tarski/Tarski–Seidenberg (1940s): the (full) L_{ring}-theories of $\mathbb{C}$, $\overline{\mathbb{F}}_p$, and $\mathbb{R}$ admit computable axiomatizations, and are thus decidable.
2. Gödel (1932): the L_{ring}-theory of $\mathbb{Z}$ admits no computable axiomatization, and thus is undecidable.
3. Robinson (1948): $\mathbb{Z}$ is definable in $\mathbb{Q}$ by an L_{ring}-formula, thus the L_{ring}-theory of $\mathbb{Q}$ is undecidable.

Three open decidability problems

- $\mathrm{Th}_\exists(\mathbb{Q})$
- $\mathrm{Th}_\exists(\mathbb{C}(t);t)$ (stronger version)
- $\mathrm{Th}(\mathbb{F}_p((t)))$

Fields of formal power series and generalized power/Hahn series:

$$F((t)) = \left\{ \sum_{n \geq N} a_n t^n \;\middle|\; a_n \in F, N \in \mathbb{Z} \right\}$$

$$F((t^\Gamma)) = \left\{ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \;\middle|\; \{\gamma : a_\gamma \neq 0\} \text{ is well-ordered} \right\}$$

For today: **Diophantine** = definable by existential formula

Examples of definable sets

— $\mathbb{R} \setminus \{0\}$ is Diophantine in $\mathbb{R}$	$xy = 1$
— $\mathbb{R}_{\geq 0}$ is Diophantine in $\mathbb{R}$	$x = y^2$
— $\mathbb{N}$ is Diophantine in $\mathbb{Z}$	$x = v^2 + w^2 + y^2 + z^2$
— $\mathbb{Q}_{\geq 0}$ is Diophantine in $\mathbb{Q}$	$0 = (x - w^2 - y^2 - z^2)(2x - w^2 - y^2 - z^2)$
— $\mathbb{Z} \setminus \{0\}$ is Diophantine in $\mathbb{Z}$	$x^2 = 1 + v^2 + w^2 + y^2 + z^2$
— $[0,1)$ is Diophantine in $\mathbb{R}$	$x = y^2/(1 + y^2)$
— $\mathbb{Z}_p$ is Diophantine in $\mathbb{Q}_p$	$1 + px^l = y^l$
— For subsets of $\mathbb{R}^n$, definable implies Diophantine	(for the o-minimalists)
— $\mathbb{Q}$ is not definable in $\mathbb{R}$	(by o-minimality)

(Alternatively, infinite Diophantine subsets of $\mathbb{R}$ contain open intervals, by henselianity!)

Preliminaries – Valued fields

A **valued field** is a pair (K, v), where K is a field and $v : K \to \Gamma \cup \{\infty\}$, for an (additive, totally) ordered abelian group Γ, such that

1. $v(x) = \infty$ iff $x = 0$,
2. $v(xy) = v(x) + v(y)$, and
3. $v(x + y) \geq \min\{v(x), v(y)\}$. Ultrametric Triangle Inequality

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- $vK = \quad$ is the **value group**
- $\mathcal{O}_v = \{x \in K \mid v(x) \quad 0\}$ is the **valuation ring**
- $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$ is the **maximal ideal**
- $Kv = \mathcal{O}_v / \mathfrak{m}_v$ is the **residue field**
- **equal characteristic** means $\mathrm{char}(K) = \mathrm{char}(Kv)$, **mixed** otherwise

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- $vK =$ is the **value group** intuition: orders of magnitude
- $\mathcal{O}_v = \{x \in K \mid v(x) \quad 0\}$ is the **valuation ring** bounded elements
- $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$ is the **maximal ideal** infinitesimals
- $Kv = \mathcal{O}_v/\mathfrak{m}_v$ is the **residue field** standard parts
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A **valued field** is a pair $(K; v)$, where K is a field and $v : K \quad [\; f \; 1 \; g$, for an (additive, totally) ordered abelian group , such that

1. $v(x) = 1$ iff $x = 0$,
2. $v(xy) = v(x) + v(y)$, and
3. $v(x + y) \quad \min f v(x); v(y) g$. Ultrametric Triangle Inequality

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- $O_v = f x \; 2 \; K \; j \; v(x) \quad 0 g$ is the **valuation ring** bounded elements
- $m_v = f x \; 2 \; K \; j \; v(x) > 0 g$ is the **maximal ideal** infinitesimals
- $Kv = O_v = m_v$ is the **residue field** standard parts
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- $(Q; v_p)$ $v_p(p^l \frac{m}{n}) = l$, for $p - m; n$ and $l; m; n \; 2 \; Z$
- $(Q_p; v_p)$ the completion
- $(F(t); v_t)$ t-adic
- $(F((t)); v_t)$ the completion

Preliminaries – Henselianity

Newton–Raphson method

(K, v) is **henselian** if for every monic $f \in \mathcal{O}_v[X]$ and every $a \in \mathcal{O}_v$ with $v(f(a)) > 0$ and $v(f'(a)) = 0$, there exists unique a' with $f(a') = 0$ and $v(a - a') > 0$.

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Theorem (Ax–Kochen/Ershov, 1965)

If (K, v) and (L, w) are henselian and of equal characteristic zero, then

$$K \equiv L \iff Kv \equiv Lw \text{ and } vK \equiv wL.$$

The existential theory of $F((t))$ – Notation

Motivated by the apparent difficulty of understanding the full $\mathsf{L}_{\mathsf{ring}}$-theories of certain fields of positive characteristic (especially henselian ones, extraspecially power series), I want to talk about an approach to understanding just their **existential theories**.

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$\mathrm{L}_{\mathrm{oag}} = \{+, -, 0, \leq, \infty\}$ — language of ordered abelian groups (with symbol for ∞).
$\mathrm{L}_{\mathrm{val}}$ — three-sorted $(\mathbf{K}, \mathbf{k}, \Gamma)$ with extra function symbols $\mathbf{v} : \mathbf{K} \to \Gamma_\infty$, $\mathbf{res} : \mathbf{K} \to \mathbf{k}$.
$\mathrm{L}_{\mathrm{val}}(\varpi)$ — expansion of $\mathrm{L}_{\mathrm{val}}$ by constant symbol ϖ.

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$\mathrm{L}_{\mathrm{val}}(\$)$ — expansion of $\mathrm{L}_{\mathrm{val}}$ by constant symbol $\$$.

F — $\mathrm{L}_{\mathrm{ring}}$-theory of fields.
OAG — $\mathrm{L}_{\mathrm{oag}}$-theory of ordered abelian groups.
$\mathbf{H}^{e0}$ — $\mathrm{L}_{\mathrm{val}}$-theory of equicharacteristic, henselian, nontrivially valued fields.
$\mathbf{H}^{e;\$}$ — $\mathrm{L}_{\mathrm{val}}(\$)$-theory of $\mathbf{H}^{e0}$ + "$\$$ interpreted by uniformizer".
$\mathbf{H}^{e;Z}$ — $\mathrm{L}_{\mathrm{val}}$-theory of $\mathbf{H}^{e0}$ + "value group is a Z-group".
$\mathbf{H}^{e}_{0}$ — $\mathrm{L}_{\mathrm{val}}$-theory of $\mathbf{H}^{e0}$ + "equal characteristic zero". Etc.

For $H = \mathbf{H}^{e0}; \mathbf{H}^{e;\$}; \mathbf{H}^{e;Z}; : : :, R \quad$ **F**, and $G \quad$ **OAG**, write

$H(R)$ for H + "residue field models R" and
$H(R; G)$ for H + "residue field models R and value group models G".

The existential theory of $F((t))$

Theorem ([DS03])

Assume Resolution of Singularities in positive characteristic. Then $\mathrm{Th}_{9}(\mathrm{F}_q((t)))$ *is decidable.*

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Theorem A ([AF16, ADF23])

For any field F:

1. $\mathrm{Th}_{\exists}(F((t)); v_t) = \mathbf{H}^{e0}(\mathrm{Th}(F))_{\exists} = \mathbf{H}^{e0}(\mathrm{Th}_{\exists}(F))_{\exists} \equiv_m \mathrm{Th}_{\exists}(F)$. NB many-one equivalence

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2. **(R4)**: $\mathrm{Th}_{\exists}(F((t)), v_t, t) = \mathbf{H}^{e,\varpi}(\mathrm{Th}(F))_{\exists} = \mathbf{H}^{e,\varpi}(\mathrm{Th}_{\exists}(F))_{\exists} \simeq_m \mathrm{Th}_{\exists}(F)$.

(R4). Every large field K (i.e. $K \preceq_{\exists} K((t))$) is existentially closed in every extension F/K for which there exists a valuation v on F/K with residue field $Fv = K$.

Importantly:

- Kuhlmann earlier showed **(R4)** holds for perfect large fields K.
- Resolution of singularities $\Longrightarrow$ Local Uniformization $\Longrightarrow$ **(R4)**.

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2. **(R4)**: $\mathrm{Th}_9(F((t)); v_t; t) = \mathbf{H}^{e;\,\$}(\mathrm{Th}(F))_9 = \mathbf{H}^{e;\,\$}(\mathrm{Th}_9(F))_9 \;{}'_m\; \mathrm{Th}_9(F)$.

Corollary ([AF16, ADF23])

1. $\mathrm{Th}_9(\mathrm{F}_q((t)); v_t) = \mathbf{H}^{e0}(\mathrm{Th}(\mathrm{F}_q))_9 = \mathbf{H}^{e0}(\mathrm{Th}_9(\mathrm{F}_q))_9 \;{}'_m\; \mathrm{Th}_9(\mathrm{F}_q)$ — decidable.
2. **(R4)**: $\mathrm{Th}_9(\mathrm{F}_q((t)); v_t; t) = \mathbf{H}^{e;\,\$}(\mathrm{Th}(\mathrm{F}_q))_9 = \mathbf{H}^{e;\,\$}(\mathrm{Th}_9(\mathrm{F}_q))_9 \;{}'_m\; \mathrm{Th}_9(\mathrm{F}_q)$ — decidable.

See also thesis of Kartas [Kar22].

Compare this with...

$$\mathrm{Th}_9(\mathrm{Q}((t)); v_t; t) = \mathbf{H}^{e;\,\$}(\mathrm{Th}(\mathrm{Q}))_9 = \mathbf{H}^{e;\,\$}(\mathrm{Th}_9(\mathrm{Q}))_9 \;{}'_m\; \mathrm{Th}_9(\mathrm{Q}) \text{ — unknown.}$$

The latter is in the territory of the classic Ax–Kochen/Ershov theorem.

Uniformities (MThms 1, 2, & 3) – A different perspective

Fragment = set of formulas closed under ^; _.

E.g. 9, 8, 89, $8_1 9$, $8^{\mathbf{k}} 9$, …

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Each of $\mathbf{H}^{e0}; \mathbf{H}^{e;\$}; \mathbf{H}^{e;Z}; : : :$ is a theory of valued fields to which we have added various complete theories on the residue field via an interpretation built naturally into the theory. In this case, the residue field is a sort and the interpretation map $_{\mathbf{k}} : \mathrm{L}_{\mathrm{ring}} \; ! \; \mathrm{L}_{\mathrm{val}}$ relativises formulas:

$$(K; v; : : :) \; 7! \; Kv$$

$$_{\mathbf{k}} \, ' \; = \; '^{\mathbf{k}} \quad [\, ' :$$

It satisfies

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Under $_{\mathbf{k}}$, existential formulas remain existential, and moreover $8_m 9_n$ becomes $8^{\mathbf{k}}_m 9^{\mathbf{k}}_n$.

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Two theories, two languages, two fragments.

valued field side			residue field side		
theory	language	fragment	theory	language	fragment
$\mathbf{H}^{e0}$	$\mathrm{L}_{\mathrm{val}}$	9	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	9
$\mathbf{H}^{e;Z}$	$\mathrm{L}_{\mathrm{val}}$	9	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	9
$\mathbf{H}^{e;\$}$	$\mathrm{L}_{\mathrm{val}}(\$)$	9	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	9

Uniformities (MThms 1, 2, & 3) – A different perspective

Let H be a theory like $\mathbf{H}^{e0}; \mathbf{H}^{e; \$}; : : :$

These settings satisfy various sensible axioms/properties:

- **Interpretations**: $(K; v; : : :) \ j= \ \ _{\mathbf{k}}' \quad , \ Kv \ j= \ \ '$.
- **Surjectivity**: for all k $j=$ $\mathbf{F}$ there exists $(K; v; : : :) \ j= \ H$ such that $\mathrm{Th}(k) \ = \mathrm{Th}(Kv)$.

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- **Surjectivity**: for all $k\ j{=}\ \mathbf{F}$ there exists $(K; v; : : :)\ j{=}\ H$ such that $\mathrm{Th}(k) = \mathrm{Th}(Kv)$.

Both results of Theorem A are deduced from axiomatizations of $\mathrm{Th}_9(K; v; : : :)$, by the background theory $H = \mathbf{H}^{e0}$ or $\mathbf{H}^{e;\$}$, together with and $_{\mathbf{k}}\mathrm{Th}_9(Kv)$. This is equivalent to

- **Monotonicity**: for all $(K; v; : : :); (L; w; : : :)\ j{=}\ H$,

$$\mathrm{Th}_9(Kv) \quad \mathrm{Th}_9(Lw) \ \ {=}\,) \ \ \mathrm{Th}_9(K; v; : : :) \quad \mathrm{Th}_9(L; w; : : :):$$

Uniformities (MThms 1, 2, & 3) – A different perspective

Let H be a theory like $\mathbf{H}^{e0}; \mathbf{H}^{e;\$}; : : :$

These settings satisfy various sensible axioms/properties:

- **Interpretations**: $(K; v; : : :) \ j\!= \ {}_{\mathbf{k}}\,' \quad , \ Kv \ j\!= \ '$.
- **Surjectivity**: for all $k \ j\!= \mathbf{F}$ there exists $(K; v; : : :) \ j\!= H$ such that $\mathrm{Th}(k) = \mathrm{Th}(Kv)$.

Both results of Theorem A are deduced from axiomatizations of $\mathrm{Th}_9(K; v; : : :)$, by the background theory $H = \mathbf{H}^{e0}$ or $\mathbf{H}^{e;\$}$, together with and ${}_{\mathbf{k}}\mathrm{Th}_9(Kv)$. This is equivalent to

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Notation/abbreviation

Write e.g. $H \ ! \ \mathbf{F}$ and $9 \ ! \ 9$ to mean we consider the theory H, the map $R \ 7\!! \ H(R)$ of theories, and existential theories on both “valued field” and “residue field” side.

Monotonicity is important for fragments like 9 that are not closed under negation.

Uniformities (MThms 1, 2, & 3) – What is the algebraic input?

Theorem B (previous results rephrased)

Monotonicity for

1. $\mathbf{H}_0^{e;\,\$}$! $\mathbf{F}_0$ and 9 ! 9 (AKE, char = 0 ‘sense check’)
2. $\mathbf{H}^{e0}$! $\mathbf{F}$ and 9 ! 9 algebraic input is [AF16]
3. **(R4)**: $\mathbf{H}^{e;\,\$}$! $\mathbf{F}$ and 9 ! 9 [ADF23]

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There is a hidden role here of *constants*: rank 0 or 1 valued subfield C, common to both structures.

Proof sketch for 3. Let $(K; v; \ _v); (L; w; \ _w)$ j= $\mathbf{H}^{e;\,\$}$. Suppose $\mathrm{Th}_9(Kv)$ $\mathrm{Th}_9(Lw)$.

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- Saturating if necessary, there exists $'_0 : Kv\ !\ Lw$.
- Extend to $\mathrm{L}_{\mathrm{val}}(\$)$-embedding $'_1 : (Kv((t));v_t;t)\ !\ (Lw((t));v_t;t)$.

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- Extend to $\mathrm{L}_{\mathrm{val}}(\$)$-embedding $'_1 : (Kv((t)); v_t; t)$! $(Lw((t)); v_t; t)$.
- Thus $\mathrm{Th}_9(Kv((t)); v_t; t)$ $\mathrm{Th}_9(Lw((t)); v_t; t)$.

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- Thus $\mathrm{Th}_9(Kv((t)); v_t; t) \quad \mathrm{Th}_9(Lw((t)); v_t; t)$.
- Let v^+ be finest proper coarsening of v. Saturating if necessary, there exists section of residue map K 99K $K \ v^+ \quad _9 \ Kv((t))$.

Uniformities (MThms 1, 2, & 3) – What is the algebraic input?

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- Thus $\mathrm{Th}_9(Kv((t)); v_t; t)$ $\mathrm{Th}_9(Lw((t)); v_t; t)$.
- Let v^+ be finest proper coarsening of v. Saturating if necessary, there exists section of residue map K 99K $K \ v^+$ $_9$ $Kv((t))$.
- Thus v^+ corresponds to $K \ v^+$-rational $K \ v^+$-place on K, so image of $K \ v^+$ in K is 9-closed in K by **(R4)**.

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- Extend to $\mathrm{L}_{\mathrm{val}}(\$)$-embedding $'_1 : (Kv((t)); v_t; t) \ ! \ (Lw((t)); v_t; t)$.
- Thus $\mathrm{Th}_9(Kv((t)); v_t; t) \quad \mathrm{Th}_9(Lw((t)); v_t; t)$.
- Let v^+ be finest proper coarsening of v. Saturating if necessary, there exists section of residue map $K \quad 99K \ K \ v^+ \quad _9 \ Kv((t))$.
- Thus v^+ corresponds to $K \ v^+$ -rational $K \ v^+$ -place on K, so image of $K \ v^+$ in K is 9-closed in K by **(R4)**.

Example

(R4): $\mathrm{Th}_9(\mathrm{F}_q((t)); v_t; t) = \mathrm{Th}_9(\mathrm{F}_q((t))((s^{\mathbb{Z}^+ \ \mathbb{Z}})); v_t \ v_s; t)$.

Uniformities (MThms 1, 2, & 3) – Consequences for decidability

Main Theorems 1,2 (“The usual corollary”—formal consequences)

1. For $\mathbf{H}_0^{e;\$}$! $\mathbf{F}_0$ and 9 ! 9:

(a) Computable elimination: $_{\$;\mathbf{k}}$: $\mathrm{Sent}_9(\mathrm{L}_{\mathrm{val}}(\$))$! $\mathrm{Sent}_9(\mathrm{L}_{\mathrm{ring}})$ such that

$$(K;v;\ _K)\ j=\ '\quad ,\quad Kv\ j=\ \ _{\$;\mathbf{k}}\,';$$

for all models $(K;v;\ _K)$ and all $'\ 2\ \mathrm{Sent}_9(\mathrm{L}_{\mathrm{val}}(\$))$.

(b) For every R $\mathrm{Sent}(\mathrm{L}_{\mathrm{ring}})$, $(\mathbf{F}_0\ [\ R)_9\ '_{\mathrm{m}}\ (\mathbf{H}_0^{e;\$}\ [\ \ _{\mathbf{k}}R)_9$.

(c) For every $(K;v;\ _K)\ j=\ \mathbf{H}_0^{e;\$}$, $\mathrm{Th}_9(K;v;\ _K) = (\mathbf{H}_0^{e;\$}\ [\ \ _{\mathbf{k}}\mathrm{Th}_9(Kv))_9$.

Uniformities (MThms 1, 2, & 3) – Consequences for decidability

Main Theorems 1,2 (“The usual corollary”—formal consequences)

1. For $\mathbf{H}_0^{e;\$}\ \ !\ \ \mathbf{F}_0$ and $9\ \ !\ \ 9$:

- **(a)** Computable elimination: $\ _{\$;\mathbf{k}}$: Sent$_9$(L$_{\text{val}}$($\$$)) $\ \ !$ Sent$_9$(L$_{\text{ring}}$) such that

$$(K;v;\ \ _K)\ j{=}\ \ '\ \ \ ,\ \ Kv\ j{=}\ \ _{\$;\mathbf{k}}\,';$$

  for all models $(K;v;\ \ _K)$ and all $'\ \ 2$ Sent$_9$(L$_{\text{val}}$($\$$)).
- **(b)** For every R Sent(L$_{\text{ring}}$), $(\mathbf{F}_0\ [\ R)_9\ \ '_{\text{m}}\ (\mathbf{H}_0^{e;\$}\ \ [\ \ _{\mathbf{k}}R)_9$.
- **(c)** For every $(K;v;\ \ _K)\ j{=}\ \mathbf{H}_0^{e;\$}$, Th$_9(K;v;\ \ _K)$ = $(\mathbf{H}_0^{e;\$}\ \ [\ \ _{\mathbf{k}}\text{Th}_9(Kv))_9$.

2. For $\mathbf{H}^{e0}\ \ !\ \ \mathbf{F}$ and $9\ \ !\ \ 9$:

- **(a)** Computable elimination: $\ _{\mathbf{k}}$: Sent$_9$(L$_{\text{val}}$) $\ \ !$ Sent$_9$(L$_{\text{ring}}$).
- **(b)** For every R Sent(L$_{\text{ring}}$), $(\mathbf{F}\ [\ R)_9\ \ '_{\text{m}}\ (\mathbf{H}^{e0}\ [\ \ _{\mathbf{k}}R)_9$.
- **(c)** For every $(K;v)\ j{=}\ \mathbf{H}^{e0}$, Th$_9(K;v)$ = $(\mathbf{H}^{e0}\ [\ \ _{\mathbf{k}}\text{Th}_9(Kv))_9$.

3. **(R4)**: For $\mathbf{H}^{e;\$}\ \ !\ \ \mathbf{F}$ and $9\ \ !\ \ 9$:

- **(a)** Computable elimination: $\ _{\$;\mathbf{k}}$: Sent$_9$(L$_{\text{val}}$($\$$)) $\ \ !$ Sent$_9$(L$_{\text{ring}}$).
- **(b)** For every R Sent(L$_{\text{ring}}$), $(\mathbf{F}\ [\ R)_9\ \ '_{\text{m}}\ (\mathbf{H}^{e;\$}\ \ [\ \ _{\mathbf{k}}R)_9$.
- **(c)** For every $(K;v;\ \ _K)\ j{=}\ \mathbf{H}^{e;\$}$, Th$_9(K;v;\ \ _K)$ = $(\mathbf{H}^{e;\$}\ \ [\ \ _{\mathbf{k}}\text{Th}_9(Kv))_9$.

Uniformities (MThms 1, 2, & 3) – Consequences for decidability

This uniform treatement yields the following example theorem, generalizing work of Sander [San96].

Main Theorem 3 ([AF26+a, Theorem 1.2])

The following theories are many-one equivalent:

1. The existential theory of Q in the language of rings.
2. The existential theory of Q((t)) in the language of rings.
3. The existential theory of Q((t)) in the language of valued fields.
4. The existential theory of Q((t)) in the language of valued fields with constant t.
5. The existential theory of large fields of characteristic zero in the language of rings.
6. The existential theory of large fields in the language of rings.
7. The existential theory of fields in the language of rings.

Proof sketch

Uniformities (MThms 1, 2, & 3) – Consequences for decidability

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6. The existential theory of large fields in the language of rings.
7. The existential theory of fields in the language of rings.

Proof sketch

1: $_{\mathrm{m}}$ 3: $_{\mathrm{m}}$ 4: is clear, and 1: $'_{\mathrm{m}}$ 4: by existential AKE in equicharacteristic zero. One can check that 2: = 5: Since each existential $\mathrm{L}_{\mathrm{ring}}$-theory of large fields of characteristic p coincides with $\mathrm{Th}_9(\mathrm{F}_p((t)))$, and these are uniformly decidable (since $\mathbf{H}^{e0}(\mathbf{F}_{>0})_9\ '_{\mathrm{m}}\ \mathrm{Th}_9(\mathbf{F}_{>0})$), we get that 5: $'_{\mathrm{m}}$ 6: Likewise 1: $'_{\mathrm{m}}$ 7: Finally 3: $_{\mathrm{m}}$ 2: using the existential and universal definitions of the valuation ring in Q((*t*)), from [AF17].

Uniformities (MThms 1, 2, & 3) – $\mathbf{H}^{e0}$! $\mathbf{F}$ and 9_n ! 9_n

For n 2 N, the 9_n fragment of the sentences of a language is the set of positive combinations of existential sentences that are in prenex form and have at most n existential quantifiers.

Theorem C ([AF26+a])

Monotonicity for $\mathbf{H}^{e0}$! $\mathbf{F}$ and 9_n ! 9_n.

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Corollary (“The usual corollary”)

For $\mathbf{H}^{e0}$! $\mathbf{F}$ *and* 9_n ! 9_n*:*

1. *Computable elimination* $_{\$;\mathbf{k}}$: $\mathrm{Sent}_{9_n}(\mathrm{L}_{\mathrm{val}}(\$))$! $\mathrm{Sent}_{9_n}(\mathrm{L}_{\mathrm{ring}})$.
2. *For every* R $\mathrm{Sent}(\mathrm{L}_{\mathrm{ring}})$, $(\mathbf{F} \ [\ R)_{9_n} \ '_{\mathrm{m}} \ (\mathbf{H}^{e0} \ [\ {}_{\mathbf{k}}R)_{9_n}$.
3. *For every* $(K; v)$ j= $\mathbf{H}^{e0}$, $\mathrm{Th}_{9_n}(K; v) = (\mathbf{H}^{e0} \ [\ {}_{\mathbf{k}}\mathrm{Th}_{9_n}(Kv))_{9_n}$.

Uniformities (MThms 1, 2, & 3) – $\mathbf{H}^{e0}$! $\mathbf{F}$ and 9_n ! 9_n

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2. *For every* R $\mathrm{Sent}(\mathrm{L}_{\mathrm{ring}})$, $(\mathbf{F} [R)_{9_n}$ $'_{\mathrm{m}}$ $(\mathbf{H}^{e0} [{}_{\mathbf{k}}R)_{9_n}$.
3. *For every* $(K;v)$ j= $\mathbf{H}^{e0}$, $\mathrm{Th}_{9_n}(K;v) = (\mathbf{H}^{e0} [{}_{\mathbf{k}}\mathrm{Th}_{9_n}(Kv))_{9_n}$.

Corollary

$\mathrm{Th}_{9_n}(F((t))) = \mathbf{H}^{e0}(\mathrm{Th}(F))_{9_n} = \mathbf{H}^{e0}(\mathrm{Th}_{9_n}(F))_{9_n}$ $'_{\mathrm{m}}$ $\mathrm{Th}_{9_n}(F)$.

In the case $F = \mathrm{F}_q$, we already knew these to be decidable.

Towards universal-existential – $\mathbf{H}^{e;Z}$! $\mathbf{F}$ and $8_1 9$! $8_1 9$

Brief sojourn into [AF25].

Theorem D ([AF25])

Monotonicity for

1. $\mathbf{H}_0^{e;\$;Z}$! $\mathbf{F}_0$ and 89 ! 89 (char = 0 sense check)
2. **(R4)**: $\mathbf{H}^{e0}$! $\mathbf{F}$ and $8^{\mathbf{k}}9$! 89
3. **(R4)**: $\mathbf{H}^{e;\$}$! $\mathbf{F}$ and $8^{\mathbf{k}}9$! 89

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Plus usual corollaries!

We obviously can't get 89 ! 89 without incorporating the value group in some way.

Towards universal-existential – $\mathbf{H}^{e;Z}$! $\mathbf{F}$ and $8_1 9$! $8_1 9$

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Plus usual corollaries!

We obviously can't get 89 ! 89 without incorporating the value group in some way.
These lemmas hold when $8_1 9$ replaced by 89 everywhere:

- $\mathrm{Th}_{89}(k) \quad \mathrm{Th}_{89}(l) \;=)\; \mathrm{Th}_{89}(k(t)^h; v_t) \quad \mathrm{Th}_{89}(l(t)^h; v_t)$
- $\mathrm{Th}_{89}(k) \quad \mathrm{Th}_{89}(l) \;=)\; \mathrm{Th}_{89}(k((t)); v_t) \quad \mathrm{Th}_{89}(l((t)); v_t)$

…It's monotonicity, Jim, but not as we know it.

Towards universal-existential – $\mathbf{H}^{e;Z}$! $\mathbf{F}$ and $8_1 9$! $8_1 9$

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…It's monotonicity, Jim, but not as we know it.

Spoiler: 89 ! 89 monotonicity fails.

Towards universal-existential – A failure of monotonicity

Definition

(K, v) is (**separably**) **defectless** if every finite (separable) extension L/K satisfies

$$\sum_{w \quad v} e(w/v) \quad f(w/v) \quad p^{d(w/v)},$$

where p is the characteristic exponent of Kv, $e(w/v)$ the ramification degree, $f(w/v)$ the inertia degree.

Let $\mathbf{HD}^{e,\$,Z}$ be the $L_{\mathrm{val}}(\$)$-theory of equicharacteristic henselian defectless valued fields (K, v, t) with $(vK, v(t)) \quad (Z, 1)$.

Theorem (Kuhlmann, [Kuh01])

$\mathbf{HD}^{e,\$,Z}(\mathrm{Th}(F_p))$ *is incomplete.*

Towards universal-existential – A failure of monotonicity

Definition

$(K;v)$ is (**separably**) **defectless** if every finite (separable) extension $L{=}K$ satisfies

$$\sum_{w \quad v} e(w{=}v) \quad f(w{=}v) \quad p^{d(w=v)};$$

where p is the characteristic exponent of Kv, $e(w{=}v)$ the ramification degree, $f(w{=}v)$ the inertia degree.

Let $\mathbf{HD}^{e;\$;\mathrm{Z}}$ be the $\mathrm{L}_{\mathrm{val}}(\$)$-theory of equicharacteristic henselian defectless valued fields $(K;v;t)$ with $(vK;v(t)) \quad (\mathrm{Z};1)$.

Theorem (Kuhlmann, [Kuh01])

$\mathbf{HD}^{e;\$;\mathrm{Z}}(\mathrm{Th}(\mathrm{F}_p))$ *is incomplete. There is an* $8_1 9$*-sentence undecided by this theory.*

Towards universal-existential – A failure of monotonicity

Definition

(K, v) is (**separably**) **defectless** if every finite (separable) extension L/K satisfies

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where p is the characteristic exponent of Kv, $e(w/v)$ the ramification degree, $f(w/v)$ the inertia degree.

Let $\mathbf{HD}^{e,\varpi,\mathbb{Z}}$ be the $\mathrm{L}_{\mathrm{val}}(\varpi)$-theory of equicharacteristic henselian defectless valued fields (K, v, t) with $(vK, v(t)) \quad (\mathbb{Z}, 1)$.

Theorem (Kuhlmann, [Kuh01])

$\mathbf{HD}^{e,\varpi,\mathbb{Z}}(\mathrm{Th}(\mathbb{F}_p))$ *is incomplete. There is an* $\forall_1\exists$*-sentence undecided by this theory.*

Corollary

Monotonicity fails for $\mathbf{HD}^{e,\varpi,\mathbb{Z}}$ and (theory) $\to \forall_1\exists$.

Towards universal-existential – A failure of monotonicity

Definition

$(K; v)$ is (**separably**) **defectless** if every finite (separable) extension $L=K$ satisfies

$$\underset{w\quad v}{X} e(w=v)\quad f(w=v)\quad p^{d(w=v)};$$

where p is the characteristic exponent of Kv, $e(w=v)$ the ramification degree, $f(w=v)$ the inertia degree.

Let $\mathbf{HD}^{e;\,\$\,;Z}$ be the $\mathrm{L}_{\mathrm{val}}(\$)$-theory of equicharacteristic henselian defectless valued fields $(K; v; t)$ with $(vK; v(t)) \quad (Z; 1)$.

Theorem (Kuhlmann, [Kuh01])

$\mathbf{HD}^{e;\,\$\,;Z}(\mathrm{Th}(\mathrm{F}_p))$ *is incomplete. There is an* $8_1 9$*-sentence undecided by this theory.*

Corollary

Monotonicity fails for $\mathbf{HD}^{e;\,\$\,;Z}$ and (theory) $\;!\;\; 8_1 9$.

So, trivially:

Monotonicity fails for $\mathbf{H}^{e;\,\$\,;Z}$ and (theory) $\;!\;\; 8_1 9$.

Towards universal-existential – Summary

valued field side			residue field side			
theory	language	fragment	theory	language	fragment	monotonicity
$\mathbf{H}_0^{e;\,\$}$	$\mathrm{L}_{\mathrm{val}}(\$)$	*9*	**F**	$\mathrm{L}_{\mathrm{ring}}$	*9*	X
$\mathbf{H}_0^{e;\,\$\,;Z}$	$\mathrm{L}_{\mathrm{val}}(\$)$	*89*	**F**	$\mathrm{L}_{\mathrm{ring}}$	*89*	X

(ic)=immediate consequence of something earlier

There are some redundancies here …

Towards universal-existential – Summary

valued field side			residue field side			
theory	language	fragment	theory	language	fragment	monotonicity
$\mathbf{H}_0^{e;\,\$}$	$L_{\mathrm{val}}(\$)$	9	**F**	L_{ring}	9	X
$\mathbf{H}_0^{e;\,\$\,;Z}$	$L_{\mathrm{val}}(\$)$	89	**F**	L_{ring}	89	X
$\mathbf{H}^{e0}$	L_{val}	9_n	**F**	L_{ring}	9_n	X
$\mathbf{H}^{e0}$	L_{val}	9	**F**	L_{ring}	9	X(ic)
$\mathbf{H}^{e0}$	L_{val}	$8^{\mathbf{k}}9$	**F**	L_{ring}	89	**(R4)**X

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valued field side			residue field side			
theory	language	fragment	theory	language	fragment	monotonicity
$\mathbf{H}_0^{e; \$}$	$L_{val}(\$)$	9	$\mathbf{F}$	L_{ring}	9	X
$\mathbf{H}_0^{e; \$;Z}$	$L_{val}(\$)$	89	$\mathbf{F}$	L_{ring}	89	X
$\mathbf{H}^{e0}$	L_{val}	9_n	$\mathbf{F}$	L_{ring}	9_n	X
$\mathbf{H}^{e0}$	L_{val}	9	$\mathbf{F}$	L_{ring}	9	X(ic)
$\mathbf{H}^{e0}$	L_{val}	$8^{\mathbf{k}}9$	$\mathbf{F}$	L_{ring}	89	**(R4)**X
$\mathbf{H}^{e; \$}$	$L_{val}(\$)$	9	$\mathbf{F}$	L_{ring}	9	**(R4)**X
$\mathbf{H}^{e; \$}$	$L_{val}(\$)$	$8^{\mathbf{k}}9$	$\mathbf{F}$	L_{ring}	89	**(R4)**X

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$\mathbf{H}_0^{e;\,\$}$	$L_{val}(\$)$	9	**F**	L_{ring}	9	X
$\mathbf{H}_0^{e;\,\$\,;Z}$	$L_{val}(\$)$	89	**F**	L_{ring}	89	X
$\mathbf{H}^{e0}$	L_{val}	9_n	**F**	L_{ring}	9_n	X
$\mathbf{H}^{e0}$	L_{val}	9	**F**	L_{ring}	9	X(ic)
$\mathbf{H}^{e0}$	L_{val}	$8^{\mathbf{k}}9$	**F**	L_{ring}	89	**(R4)**X
$\mathbf{H}^{e;\,\$}$	$L_{val}(\$)$	9	**F**	L_{ring}	9	**(R4)**X
$\mathbf{H}^{e;\,\$}$	$L_{val}(\$)$	$8^{\mathbf{k}}9$	**F**	L_{ring}	89	**(R4)**X
$\mathbf{H}^{e;Z}$	L_{val}	9	**F**	L_{ring}	9	X(ic)
$\mathbf{H}^{e;Z}$	L_{val}	$8_1 9$	**F**	L_{ring}	$8_1 9$	**(R4)**X

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valued field side			residue field side			
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$\mathbf{H}_0^{e;\,\$\,;Z}$	$\mathrm{L}_{\mathrm{val}}(\$)$	89	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	89	X
$\mathbf{H}^{e0}$	$\mathrm{L}_{\mathrm{val}}$	9_n	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	9_n	X
$\mathbf{H}^{e0}$	$\mathrm{L}_{\mathrm{val}}$	9	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	9	X(ic)
$\mathbf{H}^{e0}$	$\mathrm{L}_{\mathrm{val}}$	$8^{\mathbf{k}}9$	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	89	**(R4)**X
$\mathbf{H}^{e;\,\$}$	$\mathrm{L}_{\mathrm{val}}(\$)$	9	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	9	**(R4)**X
$\mathbf{H}^{e;\,\$}$	$\mathrm{L}_{\mathrm{val}}(\$)$	$8^{\mathbf{k}}9$	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	89	**(R4)**X
$\mathbf{H}^{e;Z}$	$\mathrm{L}_{\mathrm{val}}$	9	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	9	X(ic)
$\mathbf{H}^{e;Z}$	$\mathrm{L}_{\mathrm{val}}$	8_19	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	8_19	**(R4)**X
$\mathbf{HD}^{e;\,\$\,;Z}$	$\mathrm{L}_{\mathrm{val}}(\$)$	8_19	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	(theory)	x
$\mathbf{H}^{e;\,\$\,;Z}$	$\mathrm{L}_{\mathrm{val}}(\$)$	8_19	$\mathbf{F}$	$\mathrm{L}_{\mathrm{ring}}$	(theory)	x (ic)

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Separably tame valued fields (MThm 6)

Extending previous work of Delon and others on algebraically maximal fields satisfying Kaplansky's hypothesis, Kuhlmann introduced tame and separably tame valued fields:

Definition

We let $\mathbf{TVF}^{\mathrm{eq}}$ be the theory of **tame** valued fields of equal characteristic, i.e. the valued fields of equal characteristic that are perfect, henselian, and defectless. Let $\mathbf{STVF}^{\mathrm{eq}}$ be the theory of **separably tame** valued fields of equal characteristic. These are those valued fields $(K;v)$ of equal characteristic $p > 0$ that are henselian and separably defectless, and are such that vK is p-divisible and Kv is perfect. We denote by $\mathbf{STVF}^{\mathrm{eq}}_{\mathrm{i}}$ the theory with additional axioms to specify the imperfection degree i. Thus $\mathbf{TVF}^{\mathrm{eq}} = \mathbf{STVF}^{\mathrm{eq}}_{0}$.

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Defectlessness and separable defectlessness in certain circumstances are strong enough to provide an Ax–Kochen/Ershov Principle in positive characteristic, in the case of separably tame fields of finite imperfection degree:

Theorem ([Kuh16, KP16])

Let k be perfect of characteristic $p\ 2\ \mathrm{P}$, let   be p-divisible, and let $\mathrm{i}\ 2\ \mathrm{N}\ \ _0$. Then $\mathbf{TVF}^{\mathrm{eq}}(\mathrm{Th}(k); \mathrm{Th}(\ \ ))$ *and* $\mathbf{STVF}^{\mathrm{eq}}_{p;\mathrm{i}}(\mathrm{Th}(k); \mathrm{Th}(\ \ ))$ *are complete.*

In particular, $\mathbf{TVF}^{\mathrm{eq}}_{p}(\mathrm{Th}(k); \mathrm{Th}(\ \ ))$ is the complete theory of $(k((t^{\ \ })); v_t)$.

See recent paper of Ketelsen–Dittmann for further work on the mixed characteristic case.

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We denote by $\mathrm{L}_{\mathrm{ring},\lambda}$ the expansion of $\mathrm{L}_{\mathrm{ring}}$ by symbols for the *parametrized λ-functions*: $(a;b) \mapsto \lambda^{b}_{I}(a)$ such that

$$a = \sum b^{I} \lambda^{b}_{I}(a)^{p}$$

whenever $b \subseteq K$ is a p-independent tuple and $a \in K^{(p)}(b)$. Also $\mathrm{L}_{\mathrm{val},\lambda}$ is the corresponding expansion of $\mathrm{L}_{\mathrm{val}}$.

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whenever b K is a p-independent tuple and a 2 $K^{(p)}(b)$. Also $\mathrm{L}_{\mathrm{val};}$ is the corresponding expansion of $\mathrm{L}_{\mathrm{val}}$. Define $\mathrm{Imp}(K) = \log_{p}[K : K^{(p)}]$ if this is finite, otherwise it is simply 1 . Also

$$M \ \mathrm{V}_{C}\ N \text{ means } \mathrm{Th}(M; C) \quad \mathrm{Th}(N; C).$$

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Theorem E

Let $\mathrm{L} = \mathrm{L}_{\mathrm{val};}$ $(\mathrm{L}_{\mathbf{k}};\mathrm{L}\ )$ be a $(\mathbf{k};\ )$-expansion of $\mathrm{L}_{\mathrm{val};}$, i.e. an expansion only on the sorts $\mathrm{L}_{\mathbf{k}}$ and L . Let $K_1;K_2$ 2 $\mathbf{Mod}_{\mathrm{L}}(\mathbf{STVF}^{\mathrm{eq}})$ have common L-substructure K_0 which as a valued field is defectless, and $v_1K_1{=}v_0K_0$ is torsion-free and $K_1v_1{=}K_0v_0$ is separable.

- **(I)** $K_1 \ \mathrm{V}_{K_0} \ K_2$ in $\mathrm{Sent}_9(\mathrm{L})$ if and only if
  - **(i)** $k_1 \ \mathrm{V}_{k_0} \ k_2$ in $\mathrm{Sent}_9(\mathrm{L}_{\mathbf{k}})$,
  - **(ii)** $_1 \ \mathrm{V} \ _0 \ _2$ in $\mathrm{Sent}_9(\mathrm{L}\ )$, and
  - **(iii)** $\mathrm{Imp}(K_1{=}K_0) \quad \mathrm{Imp}(K_2{=}K_0)$.
- **(II)** $K_1 \ \mathrm{V}_{K_0} \ K_2$ in $\mathrm{Sent}(\mathrm{L})$ if and only if
  - **(i)** $k_1 \ \mathrm{V}_{k_0} \ k_2$ in $\mathrm{Sent}(\mathrm{L}_{\mathbf{k}})$,
  - **(ii)** $_1 \ \mathrm{V} \ _0 \ _2$ in $\mathrm{Sent}(\mathrm{L}\ )$, and
  - **(iii)** $\mathrm{Imp}(K_1{=}K_0) = \mathrm{Imp}(K_2{=}K_0)$.

Separably tame valued fields (MThm 6)

Two structural ingredients used consistently throughout the theory of (separably) tame valued fields.

Theorem (Generalized Stability Theorem)

Let $(F;v)$=$(K;v)$ *be a valued function field without transcendence defect, and suppose that* $(K;v)$ *is defectless. Then* $(F;v)$ *is defectless.*

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Let $(F;v)/(K;v)$ *be an immediate valued function field, and suppose that* $(K;v)$ *is tame. Then* $(F;v)$ *is contained in the henselization of* $K(x)$ *with respect to some valuation extending* v.

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The full range of “Separable” AKE principles (as in [KP16]) hold for separably tame valued fields of infinite imperfection degree.

Main Theorem 6 ([Ans26b])

The $\mathrm{L}_{\mathrm{val}}$-theory $\mathbf{STVF}^{\mathrm{eq}}$ of separably tame valued fields of equal characteristic is complete relative to the residue field and value group sorts, and the elementary imperfection degree. Moreover, the deductive closure $\mathbf{STVF}^{\mathrm{eq}}_{I}(R; G)^{`}$ is Turing equivalent to the Turing sum $R^{`} \quad {}_{T}\, G^{`}$, and also $\mathbf{STVF}^{\mathrm{eq}}_{I}(R; G)_{9}$ is many-one equivalent to R_{9}, for every theory R of fields, every consistent theory G of nontrivial ordered abelian groups, and every finite or cofinite set I of elementary imperfection degrees.

This applies for example to work of Jahnke and van der Schaaf on “separable taming”.

Mixed characteristic, finitely ramified (MThm 4)

Model theory of henselian valued fields of mixed characteristic, finitely ramified, is well developed: [AK65b, Er65, PR84, Zie72, Ers01], etc.. Includes e.g. AKE- principle. But a AKE- / $_9$ principles were lacking, though work of e.g. Basarb [Bas78] gives AKE- but with residue rings.

Indeed: AKE-9 with respect to residue field is not true, nor is 9-decidability transfer ([Dit22]).

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(Joint work with Dittmann and Jahnke.) Fix rational prime $p > 0$, and natural number e 1. Let $\mathbf{H}_e^{(0;p)}$ be the theory of henselian valued fields of mixed characteristic $(0;p)$ of initial ramification e.

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Sketch. Identify a language $\mathrm{L}_{p;e}$ $\mathrm{L}_{\mathrm{ring}}$: it has one new predicate of a certain arity. For $(K;v)\ j= \mathbf{H}_e^{(0;p)}$, expand Kv to $\mathrm{L}_{p;e}$-structure so that it *codes* the Eisenstein polynomial that generated O_v over the Cohen ring of Kv.

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Theorem F ([ADJ24])

Monotonicity for $\mathbf{H}_e^{(0;p)}$! **F** and 9 ! $9+$. (Please ask: why not $\mathbf{F}_p$?)

Plus usual corollary!

The canonical eq-char henselian val & Z-largeness (MThm 5)

K is Z-**large** if some K has a subfield E that admits a nontrivial henselian valuation v with $Ev = K$.

Theorem ([AF17])

Let $(K; v)$ *j=* $\mathbf{H}^{e0}$.

1. $\mathcal{O}_v$ *is* 8-$\mathrm{L}_{\mathrm{ring}}$*-definable in* K *(without parameters).*
2. $\mathcal{O}_u$ *is uniformly* 8-$\mathrm{L}_{\mathrm{ring}}$*-definable in all* $(L; v)$ *j=* $\mathbf{H}^{e0}(\mathrm{Th}(Kv))$ *(without parameters).*
3. Kv *is not* Z*-large.*

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It turns out that a field k is Z-large if and only if it is **existentially t-henselian**, i.e. has the same existential theory as a field that admits a nontrivial henselian valuation.

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Main Theorem 5

Let K be any field. For all non-trivial $v \in H^e(K) \setminus \{v^e_K\}$, the valued fields (K, v) share the same existential theory. Namely

$$\mathrm{Th}_{\exists}(K, v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_{\exists} = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(Kv^e_K)_{\exists, \mathrm{L}_{\mathrm{ring}}})_{\exists}.$$

In particular, each field admits at most three expansions by equal characteristic henselian nontrivial valuations, up to the equivalence of having the same existential $\mathrm{L}_{\mathrm{val}}$-theory.

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Questions are very welcome.