A NOTE ON EXISTENTIALLY T-HENSELIAN FIELDS

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ABSTRACT. A field is existentially t-henselian if it is has the same existential theory in the first-order language of rings as a field that admits a nontrivial henselian valuation. This property turns out to be equivalent to $\mathbb{Z}$-largeness, which is a property identified in previous work with Fehm, and which holds for F if and only if $tF[\![t]\!]$ is not Diophantine in $F(\!(t)\!)$, without extra constants.

In this short note, we further investigate this property in order to count the number of existential theories of henselian valuations on a given field, and to find other characterizations of existential t-henselianity.

In this note we define a field to be "existentially t-henselian" if it is has the same existential theory in the first-order language of rings as a field that admits a nontrivial henselian valuation. This property turns up naturally in the study of existential theories of equicharacteristic henselian valuations. Perhaps more surprisingly, it turn out to be also equivalent to "$\mathbb{Z}$-largeness", another property identified in previous work with Fehm, namely [AF17], and which holds for a field F if and only if $tF[\![t]\!]$ is definable in $F(\!(t)\!)$ by an existential formula in the language of rings, without parameters.

We prove several small results on existentially t-henselian fields. As an example, we are able to count the number of existential theories of henselian valuations on a given field:

Main Theorem. *Let K be any field. There are at most three existential theories $\mathrm{Th}_\exists(K, v)$ in the language of valued fields, for equicharacteristic henselian valuations v on K.*

This theorem follows from Theorem 3.4.

1. CANONICAL VALUATIONS

For each field K, let $\mathsf{S}_{\mathrm{val}}(K)$ be the partially ordered set[1] of valuations of K, where the ordering corresponds to the usual notions of coarsenings and refinements of valuations. More precisely, given valuations v, w on K, we say that v is **finer** than w (and that w is **coarser** than v), denoted $v \leq w$, if $\mathcal{O}_v \subseteq \mathcal{O}_w$.

We note the following properties:

- **(i)** $\mathsf{S}_{\mathrm{val}}(K)$ is directed upwards, over sets, i.e. every non-empty subset has a maximum element. In particular there is a maximum valuation, the trivial valuation, which is denoted v_K^{triv}.
- **(ii)** The set of coarsenings of a given valuation is a chain, i.e. a total order.
- **(iii)** Every chain has an infimum.

A valuation $v \in \mathsf{S}_{\mathrm{val}}(K)$ on K is **henselian** if for every finite extension L/K there is precisely one $w \in \mathsf{S}_{\mathrm{val}}(L)$ that restricts to v. Let $\mathsf{H}(K)$ be the partially ordered set of henselian valuations on K, and we write $\mathsf{H}_2(K) := \{v \in \mathsf{H}(K) \mid Kv \text{ is separably closed}\}$ and $\mathsf{H}_1(K) := \mathsf{H}(K) \setminus \mathsf{H}_2(K)$. Then

- $\mathsf{H}_2(K) < \mathsf{H}_1(K)$, i.e. every $v_1 \in \mathsf{H}_1(K)$ is finer than every $v_2 \in \mathsf{H}_2(K)$.
- $\mathsf{H}_1(K)$ is closed upwards, is a chain, and it has a minimum if $\mathsf{H}_2(K)$ is empty.
- $\mathsf{H}_2(K)$ is closed downwards, is not in general a chain, and it has a maximum if and only if it is nonempty.

The **canonical henselian valuation** v_K is the supremum of $\mathsf{H}_2(K)$. Note that if $\mathsf{H}_2(K) \neq \emptyset$, then $v_K = \max \mathsf{H}_2(K)$; whereas if $\mathsf{H}_2(K) = \emptyset$, then $v_K = \min \mathsf{H}_1(K)$. A field K itself is **henselian** if it admits a nontrivial henselian valuation, and it is **t-henselian** if some elementary extension $K^* \succeq K$ is henselian. It is a small part of the work of Jahnke and Koenigsmann, indeed it follows from Beth's Definability Theorem, that if Kv_K is not t-henselian, then v_K is $\mathfrak{L}_{\mathrm{ring}}$-definable without parameters. There are plenty of examples of t-henselian fields that are not henselian, including $\mathbb{R}$ and $\mathbb{F}_p^{\mathrm{alg}}$, and

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[1] This this is also known as the Riemann–Zariski space of K.

more besides. We observe also that henselianity (of a valuation) is $\mathfrak{L}_{\mathrm{val}}$-axiomatizable, but not finitely so.

A similar game can be played with p-henselian valuations, those that extend uniquely to the p-closure, as was explored by Koenigsmann, and later by Jahnke and Koenigsmann together:

Example 1.1 (p-henselian valuations, Jahnke–Koenigsmann)**.** Let $p \in \mathbb{P}$ be a prime number. For a field K, we let $\mathsf{H}^p(K)$ be the partially ordered set of p-henselian valuation rings on K. Let $\mathsf{H}^p(K)$ be the set of p-henselian valuations on K. We write $\mathsf{H}^p_1(K) = \{v \in \mathsf{H}^p(K) \mid Kv \neq Kv(p)\}$ and $\mathsf{H}^p_2(K) = \{v \in \mathsf{H}^p(K) \mid Kv = Kv(p)\}$. Then

- $\mathsf{H}^p_2(K) < \mathsf{H}^p_1(K)$, i.e. every $v_1 \in \mathsf{H}^p_2(K)$ is strictly finer than every $v_2 \in \mathsf{H}^p_1(K)$.
- $\mathsf{H}^p_1(K)$ is closed upwards, is a chain, and it has a minimum if $\mathsf{H}_2(K)$ is empty.
- $\mathsf{H}^p_2(K)$ is closed downwards, in particular under minima of chains, but is not (in general) a chain nor closed upwards.

In contrast to the situation with henselianity, p-henselianity is finitely axiomatizable. and correspondingly membership of a valuation in each of H^p_1 and H^p_2 is finitely $\mathfrak{L}_{\mathrm{val}}$-axiomatizable. The **canonical p-henselian valuation** v^p_K is the supremum of $\mathsf{H}^p_2(K)$. Note that if $\mathsf{H}^p_2(K) \neq \emptyset$, then $v^p_K = \max \mathsf{H}^p_2(K)$; whereas if $\mathsf{H}^p_2(K)$, then $v^p_K = \min \mathsf{H}^p_1(K)$. As in the henselian case, if Kv^p_K is not p-henselian then v^p_K is $\mathfrak{L}_{\mathrm{ring}}$-definable, by Beth's Definability Theorem.

Example 1.2 (Canonical tame valuation, Ketelsen–Ramello–Szewczyk)**.** Let $\mathsf{T}(K)$ be the partially ordered set of tame valuation rings on K, and let $\mathsf{T}_1(K) = \mathsf{T}(K) \cap \mathsf{H}_1(K)$ and $\mathsf{T}_2(K) = \mathsf{T}(K) \cap \mathsf{H}_2(K)$. As in the previous cases, there is an $\mathfrak{L}_{\mathrm{ring}}$-elementary class of **t-tame** fields, which are those fields k that are $\mathfrak{L}_{\mathrm{ring}}$-elementarily equivalent to a field k' admitting a nontrivial tame valuation. We denote by v^T_K the **canonical tame valuation**, which by definition is the supremum of $\mathsf{T}_2(K)$. So if $\mathsf{T}_2(K) \neq \emptyset$, then $v^T_K = \max \mathsf{T}_2(K)$, whereas if $\mathsf{T}_2(K) = \emptyset$, then $v^T_K = \min \mathsf{T}_1(K)$. Just as in the henselian case, and in contrast to the p-henselian case, tameness is not finitely axiomatizable. One case of the main theorem of [KRS24] comes from another argument by Beth's Definability Theorem applied to this context:

Theorem 1.3 ([KRS24])**.** *If Kv^T_K is not t-tame, then v^T_K is $\mathfrak{L}_{\mathrm{ring}}$-definable.*

Example 1.4. Divisible-tame valuations are those tame valuations which also have a divisible value group. The natural definition is that k is **t-divisible-tame** if there exists $k^* \equiv k$ that admits a nontrivial divisible-tame valuation. It was shown in [AJ18, Lemma 5.1] that if k is t-divisible-tame then $k \preceq k((\mathbb{Q}))$. It was also shown in [AJ18] that for each $p \in \mathbb{P} \cup \{0\}$, there exists a t-divisible-tame field k of characteristic p that is not henselian, not real closed, and not separably closed.

Similar observations hold for “canonical valuations” in many other related settings.

Remark 1.5. Many other properties of valuations that appear in the literature behave similarly. The properties henselianity, p-henselianity, tameness[2], and divisible tameness, that appear in the preceding examples, are well-behaved in several key respects. To begin with, each of these properties is $\mathfrak{L}_{\mathrm{val}}$-axiomatizable, and most are closed under composition and decomposition. Further examples of properties P for valued fields (K, v) satisfying these basic requirements are:

- v is algebraically maximal,
- v is henselian and defectless,
- v is roughly tame,
- the value group vK is roughly p-divisible,
- the value group vK is roughly divisible,
- v is defectless and a coarsening of a given valuation w.

We need not constrain our attention to $\mathfrak{L}_{\mathrm{val}}$-axiomatizability. By working in a language of bi-valued fields, we may consider structures (K, v, w) in which v coarsens w. Thus the chain H_1 is replaced by the chain of coarsenings of each w.

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[2] Tameness is not closed under composition, since the composition of a tame valuation of mixed characteristic $(0, p)$ with, for example, a discretely valued henselian valuation of equal characteristic zero is not tame. The closure of the class of tame valued fields under composition is the class of roughly tame fields.

Example 1.6. By [AJ24, Lemma 3.4], if (K, w) is NIP and of residue characteristic $p > 0$, then w has at most one coarsening v with an imperfect residue field. Beth's Definability Theorem then implies that v is definable in (K, v, w). There is also an explicit formula known, which is due to Scanlon.

All of the above are examples of the following theorem of Ketelsen, from her thesis:

Theorem 1.7 (Ketelsen, [Ket26, Theorem 4.2.4])**.** *Let P be a property of (possibly enriched) valued fields, and P^* be a property of valued fields such that*

- **(i)** *P is preserved under $\mathfrak{L}'_{\mathrm{val}}$-elementary equivalence, where $\mathfrak{L}'_{\mathrm{val}} = \mathfrak{L}' \cup \{\mathcal{O}\}$ for some enrichment $\mathfrak{L}' \supseteq \mathfrak{L}$.*
- **(ii)** *P^* is first-order axiomatizable in $\mathfrak{L}_{\mathrm{val}}$.*
- **(iii)** *For any field F and valuations u and w on F that have property P: if $u \leq w$ and $\bar{u}$ is the valuation induced by u on Fw then $(Fw, \bar{u})$ has P^*.*

Now let (K, v) be such that $v \in \mathsf{H}_1(K)$, (K, v) satisfies P and Kv is not t-henselian of type P^*. Then $\mathcal{O}_v$ is $\mathfrak{L}'$-definable.

2. Diophantine henselian valuation rings and valuation ideals

In this section we consider fields K and equicharacteristic henselian nontrivial valuations v on K, with corresponding valuation rings $\mathcal{O}$. The paper [FJ17] gives a survey of the results around the subject of $\mathfrak{L}_{\mathrm{ring}}$-definable henselian valuation rings. In [AF17], the following definition was introduced.

Definition 2.1 ([AF17, Definition 3.5])**.** A field k has **embedded residue** if there exists $k^* \equiv k$ and a nontrivial valuation v on k^* with $k^*v \to k^*$. A field k is **$\mathbb{Z}$-large** if there exists $k^* \equiv k$, a subfield $E \subseteq k^*$, and a nontrivial henselian valuation v on E with $Ev \cong k^*$.

There is an underlying duality between the pictures for existential and universal definability of Diophantine henselian valuation rings, which is somewhat twisted by the appearance of the henselian valuation in the definition of largeness.

The following theorem of [AF17] characterizes (among other things) when $F[\![t]\!]$ [respectively $tF[\![t]\!]$] is Diophantine in $F(\!(t)\!)$.

Theorem 2.2 ([AF17, Theorem 1.1])**.** *Let F be a field. Then the following are equivalent.*

- **(i)** *There is an $\exists$-$\mathfrak{L}_{\mathrm{ring}}$-formula that defines $\mathcal{O}_v$ [respectively, $\mathfrak{m}_v$] in K for some equicharacteristic henselian nontrivially valued field (K, v) with residue field F.*
- **(ii)** *There is an $\exists$-$\mathfrak{L}_{\mathrm{ring}}$-formula that defines $\mathcal{O}_v$ [respectively, $\mathfrak{m}_v$] in K for every henselian valued field (K, v) with residue field elementarily equivalent to F.*
- **(iii)** *There is no elementary extension $F \preceq F^*$ with a nontrivial valuation v on F^* for which the residue field F^*v embeds into F^* [respectively, with a nontrivial henselian valuation v on a subfield E of F^* with $Ev \cong F^*$].*

We give several negative examples, i.e. examples of $\mathcal{O}$ that are not Diophantine in K.

Example 2.3 (Negative examples)**.**

- **(i)** $\mathbb{C}[\![t]\!]$ is not Diophantine in $\mathbb{C}(\!(t)\!)$: this folkloric result is explained in [AK14, Appendix A].
- **(ii)** $\mathbb{Q}_p[\![t]\!]$ is not Diophantine in $\mathbb{Q}_p(\!(t)\!)$: this is also explained in [AK14, Appendix A].
- **(iii)** $\mathbb{R}[\![t]\!]$ is not Diophantine in $\mathbb{R}(\!(t)\!)$: this is a similar direct limit argument. Any existential formula defining $\mathbb{R}[\![t]\!]$ in $\mathbb{R}(\!(t)\!)$ must also define the (nontrivial) valuation ring $\mathbb{R}[\![t]\!]^{\mathrm{Px}}$ in the Puiseux series $\mathbb{R}(\!(t)\!)^{\mathrm{Px}}$, but this is a real closed field.

These examples generalize to the following.

Example 2.4. $\mathbb{F}[\![t]\!]$ is not Diophantine in $\mathbb{F}(\!(t)\!)$, for any algebraically closed field $\mathbb{F}$.

Proof. To see this, we again give a direct limit argument: in the algebraic closure of $\mathbb{F}(\!(t)\!)$ the unique prolongation of $\mathbb{F}[\![t]\!]$ is again nontrivial and must be defined by any existential formula defining $\mathbb{F}[\![t]\!]$ in $\mathbb{F}(\!(t)\!)$ since the algebraic closure is a direct limit of isomorphic copies of $\mathbb{F}(\!(t)\!)$. ☐

The arguments here can be easily extended to show that $F[\![t]\!]$ is not Diophantine in $F(\!(t)\!)$ whenever F is a characteristic zero t-henselian field, i.e. elementarily equivalent to one admitting a nontrivial henselian valuation. In fact, as the main theorem will show, the characteristic assumption may be removed.

Turning to positive examples, we have the following.

Example 2.5 (Positive examples)**.**

- **(i)** $\mathbb{F}_q[\![t]\!]$ is Diophantine in $\mathbb{F}_q(\!(t)\!)$ for all prime powers q ([AK14]).
- **(ii)** $F[\![t]\!]$ is Diophantine in $F(\!(t)\!)$, for F a PAC field not containing the algebraic closure of its prime subfield ([Feh15]).
- **(iii)** $\mathbb{Q}[\![t]\!]$ is Diophantine in $\mathbb{Q}(\!(t)\!)$ ([AF17]).

Each of these can be seen in a rather concrete fashion, with explicit formulas.

Remark 2.6. Strictly speaking, the framework of [AF17] was that of C-**fields**: which are fields F equipped with distinguished morphisms $C \to F$, for a given integral domain C.

3. EXISTENTIALLY T-HENSELIAN FIELDS

For a field K, we denote by $\mathcal{H}^e(K)$ the set of (equivalence classes of) equicharacteristic henselian valuations on K, partially ordered by the relation of refinement/coarsening, with largest element v^{triv}, the trivial valuation on K. Let E denote the $\mathfrak{L}_{\mathrm{ring}}$-fragment $\mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$. We say that F satisfies **(ð)**, or is **existentially t-henselian**[3], if it satisfies the equivalent conditions in the following lemma. Compare the following with [AF17, Section 6.3].

Let $\mathbf{H}^{e\prime}$ denote the $\mathfrak{L}_{\mathrm{val}}$-theory of equicharacteristic henselian nontrivially valued fields, and let $\mathbf{H}^{e\prime}(R)$ denote $\mathbf{H}^{e\prime}$ together with axioms that impose that R holds on the residue field, for any $\mathfrak{L}_{\mathrm{ring}}$-theory R. Recall that for all such R we have $\mathbf{H}^{e\prime}(R)_\exists = \mathbf{H}^{e\prime}(R_\exists)_\exists$, which is proved in [AF26].

Lemma 3.1. *For a field F, the following are equivalent.*

- **(i)** $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F(\!(t)\!))$.
- **(ii)** $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$.
- **(iii)** $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(F))_{\mathrm{E}}$.
- **(iv)** $\mathrm{Th}_\exists(F)$ *is a fixed point of the map* $T \longmapsto \mathbf{H}^{e\prime}(T)_{\mathrm{E}}$ *from the power set of* E *to itself.*
- **(v)** *There exists a henselian field F' such that* $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')$.
- **(vi)** *F is $\mathbb{Z}$-large.*
- **(vii)** $\{F' \mid \mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')\}$ *is $\mathbb{Z}$-large.*

Proof. We already know that $\mathrm{Th}_\exists(F(\!(t)\!)) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_\exists$ by [AF26, Corollary 3.19(a)(III)], thus (a, b, c, d) are equivalent. If there exists $v \in \mathcal{H}^e(F)$ non-trivial, then $\mathrm{Th}_\exists(F, v) = \mathrm{Th}_\exists(F(\!(t)\!), v \circ v_t)$, and so $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$. Next suppose that there exists henselian F' such that $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')$. Replacing F' with an elementary extension if necessary, we may assume there exists $v \in \mathcal{H}^e(F')$ non-trivial. Then $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F') = \mathbf{H}^{e\prime}(\mathrm{Th}(F'))_{\mathrm{E}} = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$, which shows that $(e) \Rightarrow (b)$. Conversely, if $\mathrm{Th}_\exists(F) = \mathbf{H}^{e\prime}(\mathrm{Th}(F))_{\mathrm{E}}$ then $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F(\!(t)\!))$. Thus $(e) \Rightarrow (a)$.

By [AF17, Proposition 6.10 (1 $\Leftrightarrow$ 4)], $\mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F(\!(t)\!))$ is equivalent to $\mathbb{Z}$-largeness, which proves $(a) \Leftrightarrow (f)$. Clearly $(f) \Rightarrow (g)$.

Suppose that for some $F_1, F_2 \in \{F' \mid \mathrm{Th}_\exists(F) = \mathrm{Th}_\exists(F')\}$ there exists $F_0 \subseteq F_2$ and a non-trivial henselian valuation w on F_0 with $F_0 w$ isomorphic to F_1. Then w is equicharacteristic and henselian, so we may embed $F_1(t)^h$ into F_2. This shows that $\mathrm{Th}_\exists(F_1) \subseteq \mathrm{Th}_\exists(F_1(t)^h) \subseteq \mathrm{Th}_\exists(F_2)$. By [AF17, Proposition 6.10 (4 $\Rightarrow$ 1)], we conclude that F is $\mathbb{Z}$-large, i.e. $(g) \Rightarrow (f)$. ☐

Lemma 3.2. *Let K be a field and $v \in \mathcal{H}^e(K)$ non-trivial. Then Kv satisfies* **(ð)** *if and only if* $\mathrm{Th}_\exists(K, v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$.

Proof. For the implication $\Rightarrow$ we have

$$\mathrm{Th}_\exists(K, v) = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_\exists = \mathbf{H}^{e\prime}(\mathbf{H}^{e\prime}(\mathrm{Th}_\exists(Kv))_{\mathrm{E}})_\exists = \mathbf{H}^{e\prime}(\mathrm{Th}_\exists(K))_\exists = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists,$$

by Lemma 3.1. For the converse, from the hypothesis $\mathrm{Th}_\exists(K, v) = \mathbf{H}^{e\prime}(\mathrm{Th}(K))_\exists$ it follows that $\mathrm{Th}_\exists(Kv) = \mathrm{Th}_\exists(K)$, and K admits a non-trivial henselian valuation, thus Kv satisfies **(ð)**. ☐

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[3] The letter ð is pronounced “eth”.

Note that any field with an algebraically closed subfield satisfies **(ð)**. We are now in a position to describe the theories $\mathrm{Th}_\exists(K,v)$ for all $v \in \mathcal{H}^e(K)$.

Theorem 3.3 (Dichotomy). *Let K be a field of characteristic $p \in \mathbb{P}\cup\{0\}$. We have the following dichotomy:*

- **(i)** *Either $\mathcal{H}^e(K)$ is not linearly ordered, in which case $\mathbb{F}_p^{\mathrm{alg}} \subseteq K$, and for all non-trivial $v \in \mathcal{H}^e(K)$ we have that Kv satisfies* **(ð)** *and*
$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(K))_\exists = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{ACF}_p)_\exists.$$
- **(ii)** *Or $\mathcal{H}^e(K)$ is linearly ordered, in which case it admits a smallest element v_K^e, and for all non-trivial $v \in \mathcal{H}^e(K) \setminus \{v_K^e\}$ we have Kv satisfies* **(ð)** *and*
$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(K))_\exists = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_\exists.$$
Moreover if v_K^e is non-trivial then $\mathrm{Th}_\exists(K,v_K^e) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(Kv_K^e))_\exists$, *and this equals* $\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(K))_\exists$ *if and only if Kv_K^e satisfies* **(ð)**.

In particular, Kv does not satisfy **(ð)** *for at most one $v \in \mathcal{H}^e(K)$. If such a v exists, it is the finest element v_K^e of $\mathcal{H}^e(K)$, which is totally ordered.*

Proof. If $\mathcal{H}^e(K)$ is not linearly ordered then there exist equicharacteristic henselian non-trivial valuations v on K with Kv separably closed. By Hensel's Lemma, $\mathbb{F}_p^{\mathrm{alg}} \subseteq K$, and thus for all $v \in \mathcal{H}^e(K)$ the residue field Kv contains $\mathbb{F}_p^{\mathrm{alg}}$, thus $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{ACF}_p)_\exists$. Moreover Kv satisfies **(ð)**, so $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(K))_\exists$ by Lemma 3.2.

Suppose that $\mathcal{H}^e(K)$ is linearly ordered. Let v_K^e denote the finest element of $\mathcal{H}^e(K)$: the valuation ring corresponding to v_K^e is the intersection of all the valuations rings of elements of $\mathcal{H}^e(K)$, and such intersections of chains of equicharacteristic henselian valuation rings are again equicharacteristic henselian valuation rings. For every non-trivial $v \in \mathcal{H}^e(K) \setminus \{v_K^e\}$, v_K^e induces a non-trivial equicharacteristic henselian valuation $\bar{v}_K^e$ on Kv. Straight away we have $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(Kv))_\exists$ and $\mathrm{Th}_\exists(Kv,\bar{v}_K^e) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(Kv_K^e))_\exists$. Combining these equalities we have $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_\exists$. Moreover Kv satisfies **(ð)**, so $\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(K))_\exists$ by Lemma 3.2.

Finally we suppose that v_K^e is non-trivial. Then straight away we have $\mathrm{Th}_\exists(K,v_K^e) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(Kv_K^e))_\exists$. It only remains to argue that Kv_K^e satisfies **(ð)** if and only if $\mathrm{Th}_\exists(K,v_K^e) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(K))_\exists$, but this is simply Lemma 3.2 applied to $v = v_K^e$. ☐

We can rephrase this theorem as follows:

Theorem 3.4. *Let K be any field. For all non-trivial $v \in \mathcal{H}^e(K) \setminus \{v_K^e\}$, the valued fields (K,v) share the same existential theory. Namely:*
$$\mathrm{Th}_\exists(K,v) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(K))_\exists = \mathbf{H}^{\mathrm{e}\prime}(\underbrace{\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(k))^{\vdash} \cap \mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}_{\text{the deductive closure in } \mathfrak{L}_{\mathrm{ring}} \text{ of } \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(k))})_\exists,$$
where k denotes the residue field of v_K^e.

This theorem permits several rephrasings and corollaries.

Corollary 3.5. *In case* **(i)** *of the theorem, all non-trivial $v \in \mathcal{H}^e(K)$ share the same existential theory, namely* $\mathbf{H}^{\mathrm{e}\prime}(\mathbf{ACF}_p)_\exists$. *In case* **(ii)** *of the theorem, all non-trivial $v \in \mathcal{H}^e(K)$ not equal to v_K^e share the same existential theory, namely* $\mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}(Kv_K^e))_{\mathrm{E}})_\exists$.

Corollary 3.6. *For any field K, there are at most three distinct existential theories of the valued fields (K,v), for $v \in \mathcal{H}^e(K)$:*

- **(I)** *the trivial case $v = v_{\mathrm{triv}}$,*
- **(II)** *v is non-trivial and Kv satisfies* **(ð)**,
- **(III)** *v is non-trivial and Kv does not satisfy* **(ð)**.

Cases **(I,II,III)** *are always distinct. Case* **(I)** *exists for every K, but both cases* **(II)** *and* **(III)** *may be void, independently, according to the following examples. However, since at most one element of $\mathcal{H}^e(K)$ satisfies* **(III)**, *case* **(II)** *is void if and only if either $\mathcal{H}^e(K) = \{v^{\mathrm{triv}}\}$ or $\mathcal{H}^e(K) = \{v^{\mathrm{triv}}, v_K^e\}$ with Kv_K^e not satisfying* **(ð)**.

The Main Theorem is also a corollary of these results.

Example 3.7.

(a) Let $K = \mathbb{Q}$. There are no non-trivial elements of $\mathcal{H}^e(\mathbb{Q})$. In this case $\mathbb{Q}$ does not satisfy **(ð)**. Both (II) and (III) are void.

(b) Let $K = \mathbb{Q}^{\mathrm{alg}}$. There are no non-trivial elements of $\mathcal{H}^e(\mathbb{Q}^{\mathrm{alg}})$. In this case $\mathbb{Q}^{\mathrm{alg}}$ satisfies **(ð)**. Nevertheless, both (II) and (III) are void.

(c) Let $K = \mathbb{Q}(\!(t)\!)$. The only non-trivial element of $\mathcal{H}^e(\mathbb{Q}(\!(t)\!))$ is v_t, which is in case (III) since $\mathbb{Q}$ does not satisfy **(ð)**. Thus (II) is void.

(d) Let $K = \mathbb{Q}^{\mathrm{alg}}(\!(t)\!)$. The only non-trivial element of $\mathcal{H}^e(\mathbb{Q}^{\mathrm{alg}}(\!(t)\!))$ is v_t, which is in case (II) since $\mathbb{Q}^{\mathrm{alg}}$ satisfies **(ð)**. Thus (III) is void.

(e) Let $K = \mathbb{Q}(\!(s)\!)(\!(t)\!)$. There are two non-trivial elements of $\mathcal{H}^e(\mathbb{Q}(\!(s)\!)(\!(t)\!))$, namely v_t, which is in case (II), and $v_s \circ v_t$, which is in case (III). Note that $\mathbb{Q}$ and $\mathbb{Q}(\!(s)\!)$ have different existential theories, thus so do $(\mathbb{Q}(\!(s)\!)(\!(t)\!), v_t)$ and $(\mathbb{Q}(\!(s)\!)(\!(t)\!), v_s \circ v_t)$.

Let $A = (B, \hat{B}, \iota)$ be an arch. Suppose ($\star$) that $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$, $\hat{L}_1 = \mathrm{Sent}(\mathfrak{L}_1) \cap \hat{L}_2$, and $L_1 = \mathrm{Sent}(\mathfrak{L}_1) \cap L_2$. We do not suppose ι to be the identity map. The set of L_1-deductively closed L_1-theories forms a complete lattice. Consider the map from $\mathfrak{L}_1$-theories to L_1-theories given by $R \mapsto T_2(R)_{L_1}$, and let Φ_A denote the restriction of this map to L_1-theories.

Definition 3.8. We say that Φ_A is **increasing** if $R \subseteq \Phi_A(R)$, for all L_1-theories R; and that Φ_A is **idempotent** if $\Phi_A(R) = \Phi_A \circ \Phi_A(R)$, for all R.

It is easy to see that $R \subseteq S$ implies $\Phi_A(R) \subseteq \Phi_A(S)$. Now we consider the arch $A = \mathrm{F}_\exists/\mathrm{H}^{\mathrm{e}\prime}_\exists \parallel \mathrm{F}/\mathrm{H}^{\mathrm{e}\prime}$. Note that the hypotheses ($\star$) are satisfied for A. We continue to denote $\mathrm{E} = L_1 = \mathrm{Sent}_\exists(\mathfrak{L}_{\mathrm{ring}})$ and note that in this case $\iota = \iota_{\mathbf{k}}$ is not the identity map. By Lemma 3.1, a theory $\mathrm{Th}_\exists(k)$ is a fixed point of Φ_A if and only if k satisfies **(ð)**.

It is easy to see that Φ_A is increasing, at least when $R = \mathrm{Th}_\exists(k)$ is the existential theory of a field: since $k \subseteq k(\!(t)\!)$ and $\mathrm{Th}_\exists(k(\!(t)\!)) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k))_{\mathrm{E}}$, we have $\mathrm{Th}_\exists(k) \subseteq \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(k))_{\mathrm{E}}$.

Lemma 3.9. *Φ_A is increasing.*

Proof. Let R be an E-theory, so that $R = R_\exists$. We want $R \subseteq \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}}$, i.e. $\mathbf{H}^{\mathrm{e}\prime}(R) \models R$. Let $(K, v) \models \mathbf{H}^{\mathrm{e}\prime}(R)$. Then $Kv \models R$. By [ADF23] there exists an elementary extension $(K, v) \preceq (K^*, v^*)$ and a partial section $Kv \to K^*$ of the residue map of v^*. Thus $\mathrm{Th}_\exists(Kv) \subseteq \mathrm{Th}_\exists(K^*, v^*)$. Thus $(K^*, v^*) \models R$. ☐

Lemma 3.10. $\mathrm{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$ *for each $\mathfrak{L}_{\mathrm{ring}}$-theory R.*

Proof. Since Φ_A is increasing we have $R \subseteq \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. Thus $\mathbf{H}^{\mathrm{e}\prime}(R) \subseteq \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})$. Let $(K, v) \models \mathbf{H}^{\mathrm{e}\prime}(R)$, so that $Kv \models R$. There exists an elementary extension $(K, v) \preceq (K^*, v^*)$ such that v^* admits a nontrivial proper coarsening w. Then $K^*v^* \models R$, so $(K^*w, barv) \models \mathbf{H}^{\mathrm{e}\prime}(R)$ and $K^*w \models \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. Therefore $(K^*, w) \models \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})$, and so $K^* \models \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. This proves that $\mathbf{H}^{\mathrm{e}\prime}(R) \models \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})}$. ☐

Lemma 3.11. *Φ_A is idempotent.*

Proof. Let R be an E-theory. Taking the existential consequences of the previous lemma we have $\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}} = \mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{Sent}(\mathfrak{L}_{\mathrm{ring}})})_{\mathrm{E}}$. The latter is equal to $\mathbf{H}^{\mathrm{e}\prime}(\mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}})_{\mathrm{E}}$. ☐

Question 3.12. What can we say about $\mathfrak{L}_{\mathrm{ring}}$-theories R that are fixed points of Φ_A?

Observation 3.13. *Let R be an* E*-theory of fields. The following are equivalent.*

(a) *R is a fixed point of Φ_A.*

(b) *R is in the image of Φ_A.*

(c) *Every model of R is $\mathbb{Z}$-large and $R = R_\exists$.*

Proof. The equivalence between (a) and (b) is trivial since Φ_A is idempotent. For (c) $\Rightarrow$ (a): by monotonicity we have $R \subseteq \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}}$. If $F \models R$ then F is $\mathbb{Z}$-large, so $\mathrm{Th}_\exists(F) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(F))_{\mathrm{E}}$ by Lemma 3.1. Therefore $\mathrm{Th}_\exists(F) = \mathbf{H}^{\mathrm{e}\prime}(\mathrm{Th}_\exists(F))_{\mathrm{E}} \supseteq \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}}$, and so $R \models \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}}$. Since also $R = R_{\mathrm{E}}$, we have $R_{\mathrm{E}} = \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}}$, which proves ($a$). Next, suppose ($a$) and let $F \models R = \mathbf{H}^{\mathrm{e}\prime}(R)_{\mathrm{E}}$. Then F is a model of the existential $\mathfrak{L}_{\mathrm{ring}}$-theory of a henselian field, thus it is $\mathbb{Z}$-large. ☐

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REFERENCES

[ADF23] S. Anscombe, P. Dittmann, and A. Fehm. Axiomatizing the existential theory of $\mathbb{F}_q(\!(t)\!)$. To appear in *Algebra & Number Theory*, 2023. arXiv:2205.05438 [math.LO]

[AF16] S. Anscombe and A. Fehm. The existential theory of equicharacteristic henselian valued fields. *Algebra & Number Theory*, 10-3, 665–683, 2016.

[AF17] S. Anscombe and A. Fehm. Characterizing diophantine henselian valuation rings and valuation ideals. *Proc. Lond. Math. Soc.*, 115:293–322, 2017.

[AF26] S. Anscombe and A. Fehm. Interpretations of syntactic fragments of theories of fields. To appear in *Israel Journal of Mathematics*, 2026. arXiv:2312.17616 [math.LO]

[AJ18] S. Anscombe and F. Jahnke. Henselianity in the language of rings. *Annals of Pure and Applied Logic*, 169(9):872–895, 2018.

[AJ24] S. Anscombe and F. Jahnke. Characterizing NIP henselian fields. *J. Lond. Math. Soc.*, (2) 109 (2024), no. 3.

[AK14] S. Anscombe and J. Koenigsmann. An existential $\emptyset$-definition of $\mathbb{F}_q[\![t]\!]$ in $\mathbb{F}_q(\!(t)\!)$. *J. Symb. Log.* 79 (2014), no. 4, 1336–1343.

[CDLM13] R. Cluckers, J. Derakhshan, E. Leenknegt, and A. Macintyre. Uniformly defining valuation rings in henselian valued fields with finite and pseudo-finite residue field. *Annals of Pure and Applied Logic*, vol. 164 (2013), pp. 1236–1246.

[EP00] A.J. Engler and A. Prestel. *Valued Fields.* Springer, Monographs in Mathematics, 2005.

[Feh15] A. Fehm. Existential $\emptyset$-definability of Henselian valuation rings. *J. Symb. Log.*, 80 (2015), no. 1, 301–307.

[FJ17] A. Fehm and F. Jahnke. Recent progress on definability of Henselian valuations. In *Ordered algebraic structures and related topics*, 135–143. Contemp. Math., 697 American Mathematical Society, Providence, RI, 2017.

[Ket26] M. Ketelsen. Understanding the model theory of henselian valued fields by parts. PhD Thesis, University of Münster, 2026.

[KRS24] M. Ketelsen, S. Ramello, and P. Szewczyk Definable henselian valuations in positive residue characteristic. *Journal of Symbolic Logic*, published online, 2024.

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