What Model companionship can say about the Continuum problem

Matteo Viale

Dipartimento di Matematica
Università di Torino

Logique à Paris – Paris – 9/5/2023

Section 1

Basics on the Continuum problem

Given sets X, Y

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Given sets X, Y

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Cardinal arithmetic in a nutshell

- $|X| \leq |Y|$ and $|Y| \leq |X|$ iff $|X| = |Y|$ (Cantor 1887, Bernstein 1897, Dedekind 1898).

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Cardinal arithmetic in a nutshell

- $|X| \leq |Y|$ and $|Y| \leq |X|$ iff $|X| = |Y|$ (Cantor 1887, Bernstein 1897, Dedekind 1898).
  - $|[0;1]| \leq |(0;1)|$ and $|[0;1]| \geq |(0;1)|$ witnessed by continuous functions.
  - $f : [0;1] \to (0;1)$ bijection, f is not continuous.

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Cardinal arithmetic in a nutshell

- $|X| \leq |Y|$ and $|Y| \leq |X|$ iff $|X| = |Y|$ (Cantor 1887, Bernstein 1897, Dedekind 1898).
- $|X| < |\mathcal{P}(X)|$ (Cantor 1891).

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Cardinal arithmetic in a nutshell

- $|X| \leq |Y|$ and $|Y| \leq |X|$ iff $|X| = |Y|$ (Cantor 1887, Bernstein 1897, Dedekind 1898).
- $|X| < |\mathcal{P}(X)|$ (Cantor 1891).
  If $g : X \to \mathcal{P}(X)$, g is not a surjection as witnessed by

$$Y_g = \{x \in X : x \notin g(x)\}.$$

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Cardinal arithmetic in a nutshell

- $|X| \leq |Y|$ and $|Y| \leq |X|$ iff $|X| = |Y|$ (Cantor 1887, Bernstein 1897, Dedekind 1898).
- $|X| < |\mathcal{P}(X)|$ (Cantor 1891).
- $\leq$ is a well-order on cardinals (Zermelo+... $\sim$ 1904).

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Cardinal arithmetic in a nutshell

- $|X| \leq |Y|$ and $|Y| \leq |X|$ iff $|X| = |Y|$ (Cantor 1887, Bernstein 1897, Dedekind 1898).
- $|X| < |\mathcal{P}(X)|$ (Cantor 1891).
- $\leq$ is a well-order on cardinals (Zermelo+... $\sim$ 1904), i.e. it is a linear order on cardinals such that for every class $C \neq \emptyset$ there is $\min\{|X| : X \in C\}$.

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$.

Cardinal arithmetic in a nutshell

- $|X| \leq |Y|$ and $|Y| \leq |X|$ iff $|X| = |Y|$ (Cantor 1887, Bernstein 1897, Dedekind 1898).
- $|X| < |\mathcal{P}(X)|$ (Cantor 1891).
- $\leq$ is a well-order on cardinals (Zermelo+… $\sim$ 1904).

Cardinals

- $\aleph_0 = |\mathbb{N}|$;
- $\aleph_1 = \aleph_0^+ = \min\{|Z| : |Z| > \aleph_0\}$;
- $2^{\aleph_0} = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

Continuum Hypothesis CH (Cantor 1878, Hilbert 1900)

- $\aleph_1 = 2^{\aleph_0}$, or equivalently
- if $Z \subseteq \mathbb{R}$, either $|Z| = |\mathbb{R}|$ or $|Z| \leq |\mathbb{N}|$.

Counterexamples to CH?

- No *closed* subset of $\mathbb{R}$ is a counterexample to CH (Cantor 1883).
- No *Borel* subset of $\mathbb{R}$ is a counterexample to CH (Alexandroff 1916, Hausdorff 1917).
- No *analytic* subset of $\mathbb{R}$ is a counterexample to CH (Suslin+Alexandroff 1917).

Continuum Hypothesis CH (Cantor 1878, Hilbert 1900)

- $\aleph_1 = 2^{\aleph_0}$, or equivalently
- if $Z \subseteq \mathbb{R}$, either $|Z| = |\mathbb{R}|$ or $|Z| \leq |\mathbb{N}|$.

Counterexamples to CH?

- No *closed* subset of $\mathbb{R}$ is a counterexample to CH (Cantor 1883).
- No *Borel* subset of $\mathbb{R}$ is a counterexample to CH (Alexandroff 1916, Hausdorff 1917).
- No *analytic* subset of $\mathbb{R}$ is a counterexample to CH (Suslin+Alexandroff 1917).

Continuum Hypothesis CH (Cantor 1878, Hilbert 1900)

- $\aleph_1 = 2^{\aleph_0}$, or equivalently
- if $Z \subseteq \mathbb{R}$, either $|Z| = |\mathbb{R}|$ or $|Z| \leq |\mathbb{N}|$.

Counterexamples to CH?

- No *closed* subset of $\mathbb{R}$ is a counterexample to CH (Cantor 1883).
- No *Borel* subset of $\mathbb{R}$ is a counterexample to CH (Alexandroff 1916, Hausdorff 1917).
- No *analytic* subset of $\mathbb{R}$ is a counterexample to CH (Suslin+Alexandroff 1917).

Continuum Hypothesis CH (Cantor 1878, Hilbert 1900)

- $\aleph_1 = 2^{\aleph_0}$, or equivalently
- if $Z \subseteq \mathbb{R}$, either $|Z| = |\mathbb{R}|$ or $|Z| \leq |\mathbb{N}|$.

Counterexamples to CH?

- No *closed* subset of $\mathbb{R}$ is a counterexample to CH (Cantor 1883).
- No *Borel* subset of $\mathbb{R}$ is a counterexample to CH (Alexandroff 1916, Hausdorff 1917).
- No *analytic* subset of $\mathbb{R}$ is a counterexample to CH (Suslin+Alexandroff 1917).

Continuum Hypothesis CH (Cantor 1878, Hilbert 1900)

- $\aleph_1 = 2^{\aleph_0}$, or equivalently
- if $Z \subseteq \mathbb{R}$, either $|Z| = |\mathbb{R}|$ or $|Z| \leq |\mathbb{N}|$.

Counterexamples to CH?

- No *closed* subset of $\mathbb{R}$ is a counterexample to CH (Cantor 1883).
- No *Borel* subset of $\mathbb{R}$ is a counterexample to CH (Alexandroff 1916, Hausdorff 1917).
- No *analytic* subset of $\mathbb{R}$ is a counterexample to CH (Suslin+Alexandroff 1917).

Continuum Hypothesis CH (Cantor 1878, Hilbert 1900)

- $\aleph_1 = 2^{\aleph_0}$, or equivalently
- if $Z \subseteq \mathbb{R}$, either $|Z| = |\mathbb{R}|$ or $|Z| \leq |\mathbb{N}|$.

Counterexamples to CH?

- No *closed* subset of $\mathbb{R}$ is a counterexample to CH (Cantor 1883).
- No *Borel* subset of $\mathbb{R}$ is a counterexample to CH (Alexandroff 1916, Hausdorff 1917).
- No *analytic* subset of $\mathbb{R}$ is a counterexample to CH (Suslin+Alexandroff 1917).

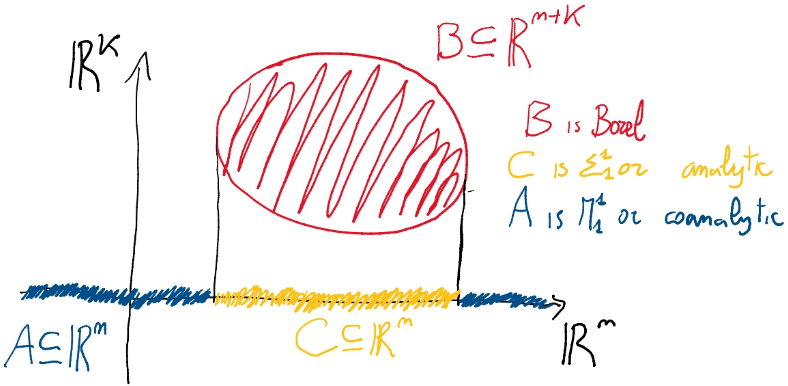

Figure: Analytic and coanalytic sets

The projective susets of $\mathbb{R}^n$ are those subsets of $\mathbb{R}^n$ which are Σ^1_m (or Π^1_m) for some m.

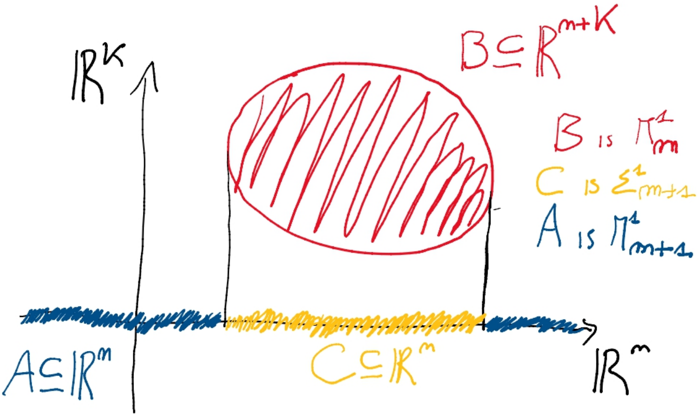

Figure: Projective sets

Counterexamples to CH? continued

Assume there is a proper class of Woodin cardinals. Then:

- No *universally Baire* subset of $\mathbb{R}$ is a counterexample to CH (Feng-Magidor-Woodin 1992 + Steel-Martin 1989 + Davis 1964).
- Borel sets, analytic sets, projective sets,. . . are all universally Baire (Feng-Magidor-Woodin 1992 + Steel-Martin 1989).

Definition

$U \subseteq \mathbb{R}$ is universally Baire if $f^{-1}[U]$ has the Baire property in X for any continuous $f : X \to \mathbb{R}$ with (X, τ) compact Hausdorff.

- Analytic and coanalytic sets are provably universally Baire without large cardinals.
- Games with payoff a universally Baire set are *determined* if (and in a weak sense only if) there is a proper class of Woodin cardinals.

Counterexamples to CH? continued

Assume there is a proper class of Woodin cardinals. Then:

- No *universally Baire* subset of $\mathbb{R}$ is a counterexample to CH (Feng-Magidor-Woodin 1992 + Steel-Martin 1989 + Davis 1964).
- Borel sets, analytic sets, projective sets,... are all universally Baire (Feng-Magidor-Woodin 1992 + Steel-Martin 1989).

Definition

$U \subseteq \mathbb{R}$ is universally Baire if $f^{-1}[U]$ has the Baire property in X for any continuous $f : X \to \mathbb{R}$ with (X, τ) compact Hausdorff.

- Analytic and coanalytic sets are provably universally Baire without large cardinals.
- Games with payoff a universally Baire set are *determined* if (and in a weak sense only if) there is a proper class of Woodin cardinals.

Counterexamples to CH? continued

Assume there is a proper class of Woodin cardinals. Then:

- No *universally Baire* subset of $\mathbb{R}$ is a counterexample to CH (Feng-Magidor-Woodin 1992 + Steel-Martin 1989 + Davis 1964).
- Borel sets, analytic sets, projective sets,... are all universally Baire (Feng-Magidor-Woodin 1992 + Steel-Martin 1989).

Definition

$U \subseteq \mathbb{R}$ is universally Baire if $f^{-1}[U]$ has the Baire property in X for any continuous $f : X \to \mathbb{R}$ with (X, τ) compact Hausdorff.

- Analytic and coanalytic sets are provably universally Baire without large cardinals.
- Games with payoff a universally Baire set are *determined* if (and in a weak sense only if) there is a proper class of Woodin cardinals.

Counterexamples to CH? continued

Assume there is a proper class of Woodin cardinals. Then:

- No *universally Baire* subset of $\mathbb{R}$ is a counterexample to CH (Feng-Magidor-Woodin 1992 + Steel-Martin 1989 + Davis 1964).
- Borel sets, analytic sets, projective sets,... are all universally Baire
  (Feng-Magidor-Woodin 1992 + Steel-Martin 1989).

Definition

$U \subseteq \mathbb{R}$ is universally Baire if $f^{-1}[U]$ has the Baire property in X for any continuous $f : X \to \mathbb{R}$ with (X, τ) compact Hausdorff.

- Analytic and coanalytic sets are provably universally Baire without large cardinals.
- Games with payoff a universally Baire set are *determined* if (and in a weak sense only if) there is a proper class of Woodin cardinals.

Counterexamples to CH? continued

Assume there is a proper class of Woodin cardinals. Then:

- No *universally Baire* subset of $\mathbb{R}$ is a counterexample to CH (Feng-Magidor-Woodin 1992 + Steel-Martin 1989 + Davis 1964).
- Borel sets, analytic sets, projective sets,. . . are all universally Baire
  (Feng-Magidor-Woodin 1992 + Steel-Martin 1989).

Definition

$U \subseteq \mathbb{R}$ is universally Baire if $f^{-1}[U]$ has the Baire property in X for any continuous $f : X \to \mathbb{R}$ with (X, τ) compact Hausdorff.

- Analytic and coanalytic sets are provably universally Baire without large cardinals.
- Games with payoff a universally Baire set are *determined* if (and in a weak sense only if) there is a proper class of Woodin cardinals.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.

Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.

Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.

This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.

Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.

Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.

Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.

This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.

Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.

Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.

Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.

This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.

Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.
Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.
Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.
This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.
Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.
Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.
Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.
This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.
Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0;1]$. It is meager.
Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.
Seen as a subset of $[0;1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0;1]$.
This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.
Hence $P \subseteq [0;1]$ is not universally Baire, even if it has the Baire property.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.
Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.
Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.
This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.
Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.
Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.
Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.
This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.
Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

Independence of CH

CH is independent of the axioms of set theory:

- There is a model of the axioms of MK where CH holds (Gödel 1939).
- There is a model of the axioms of MK where CH fails (Cohen 1963).
- In the model of the axioms of MK where CH fails produced by Cohen, this failure can be witnessed by a Σ^1_2-set of reals.

Independence of CH

CH is independent of the axioms of set theory:

- There is a model of the axioms of MK where CH holds (Gödel 1939).
- There is a model of the axioms of MK where CH fails (Cohen 1963).
- In the model of the axioms of MK where CH fails produced by Cohen, this failure can be witnessed by a Σ^1_2-set of reals.

Independence of CH

CH is independent of the axioms of set theory:

- There is a model of the axioms of MK where CH holds (Gödel 1939).
- There is a model of the axioms of MK where CH fails (Cohen 1963).
- In the model of the axioms of MK where CH fails produced by Cohen, this failure can be witnessed by a Σ^1_2-set of reals.

Independence of CH

CH is independent of the axioms of set theory:

- There is a model of the axioms of MK where CH holds (Gödel 1939).
- There is a model of the axioms of MK where CH fails (Cohen 1963).
- In the model of the axioms of MK where CH fails produced by Cohen, this failure can be witnessed by a Σ^1_2-set of reals.

Section 2

Gödel's program


WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

The American Matematical Monthly, 54(9), 1947

WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

On the undecidability of CH:

> Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality;

WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

On Large Cardinals:

For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set[17] on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of." These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers or (which is the same) of sets having these cardinal numbers. The simplest of these strong "axioms of infinity" assert the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $>\aleph_0$.

WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

On success as a criterion to detect new axioms:

> There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.

Section 3

Large cardinals

Large cardinal axioms

- Large cardinals formalize the idea that **the universe of sets is as tall as possible** i.e. the well-ordering on the cardinals is as long as possible.

Large cardinal axioms

- Large cardinals formalize the idea that **the universe of sets is as tall as possible** i.e. the well-ordering on the cardinals is as long as possible.
- Gödel already mentioned *inaccessibility*.
- Wiles proof of Fermat's last theorem uses *Grothendieck universes*.

Large cardinal axioms

- Large cardinals formalize the idea that **the universe of sets is as tall as possible** i.e. the well-ordering on the cardinals is as long as possible.
- Gödel already mentioned *inaccessibility*.
- Wiles proof of Fermat's last theorem uses *Grothendieck universes*.
- The existence of a Grothendieck universe is equivalent to the existence of an *inaccessible cardinal*.

Large cardinal axioms

- Large cardinals formalize the idea that **the universe of sets is as tall as possible** i.e. the well-ordering on the cardinals is as long as possible.
- Gödel already mentioned *inaccessibility*.
- Wiles proof of Fermat's last theorem uses *Grothendieck universes*.
- The existence of a Grothendieck universe is equivalent to the existence of an *inaccessible cardinal*.
- The existence of arbitrarily many Grothendieck universes is equivalent to:

there is a proper class of inaccessible cardinals.

Large cardinal axioms

- Large cardinals formalize the idea that **the universe of sets is as tall as possible** i.e. the well-ordering on the cardinals is as long as possible.
- Gödel already mentioned *inaccessibility*.
- Wiles proof of Fermat's last theorem uses *Grothendieck universes*.
- The existence of arbitrarily many Grothendieck universes is equivalent to:

there is a proper class of inaccessible cardinals.

Vopenka's principle

For every *proper class* of **directed graphs with no loops**, there are two members of the class with a homomorphism between them.

Large cardinal axioms

- Large cardinals formalize the idea that **the universe of sets is as tall as possible** i.e. the well-ordering on the cardinals is as long as possible.
- Gödel already mentioned *inaccessibility*.
- Wiles proof of Fermat's last theorem uses *Grothendieck universes*.
- The existence of arbitrarily many Grothendieck universes is equivalent to:

  there is a proper class of inaccessible cardinals.

Vopenka's principle

For every *proper class* of **directed graphs with no loops**, there are two members of the class with a homomorphism between them.

Adamek-Rosicky, *Locally presentable and accessible categories*, CUP, 1994.

Vopenka's principle VP

For every proper class of directed graphs with no loops, there are two members of the class with a homomorphism between them.

Vopenka's principle VP

For every proper class of directed graphs with no loops, there are two members of the class with a homomorphism between them.

Fact

Assume Vopenka's principle. Then there is a proper class of Woodin cardinals.

Vopenka's principle VP

For every proper class of directed graphs with no loops, there are two members of the class with a homomorphism between them.

From nLab:

The implication of VP on homotopy theory, model categories and cohomology localization are discussed in the following articles

- Jiří Rosický, Walter Tholen, *Left-determined model categories and universal homotopy theories* Transactions of the American Mathematical Society Vol. 355, No. 9 (Sep., 2003), pp. 3611-3623 (JSTOR).
- Carles Casacuberta, Dirk Scevenels, Jeff Smith, *Implications of large-cardinal principles in homotopical localization* Advances in Mathematics Volume 197, Issue 1, 20 October 2005, Pages 120-139
- Joan Bagaria, Carles Casacuberta, Adrian Mathias, Jiří Rosicky *Definable orthogonality classes in accessible categories are small*, arXiv
- Giulio Lo Monaco, *Vopěnka's principle in ∞-categories*, arxiv:2105.04251

JOURNALS » JEMS » VOL. 17, NO. 3

Journal of the European Mathematical Society

Volume 17, No. 3 (2015)

Section 4

Forcing axioms

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

Forcing axioms relative to a cardinal κ:

The powerset of X is “as thick as possible” for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

The rest of the talk is mainly aimed at formulating precisely the second of these two concepts.

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.
- MM^{++} and $(*)$ are forcing axioms for $\aleph_1$ the first uncountable cardinal.
- Baire's category theorem is a "topological" forcing axiom for $\aleph_0$.
- Large cardinals entail "algebraic" forcing axioms for $\aleph_0$.

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.
- MM^{++} and $(*)$ are forcing axioms for $\aleph_1$ the first uncountable cardinal.
- Baire's category theorem is a "topological" forcing axiom for $\aleph_0$.
- Large cardinals entail "algebraic" forcing axioms for $\aleph_0$.

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.
- MM^{++} and $(*)$ are forcing axioms for $\aleph_1$ the first uncountable cardinal.
- Baire's category theorem is a "topological" forcing axiom for $\aleph_0$.
- Large cardinals entail "algebraic" forcing axioms for $\aleph_0$.

Section 5

Algebraic closure and model companionship

Algebraic closure of structures for $\{+, \cdot, 0, 1\}$

Structures	Axioms	Example
Commutative semirings with no zero divisors	$\forall x, y\,(x \cdot y = y \cdot x)$ $\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ $\forall x\,(x \cdot 1 = x \wedge 1 \cdot x = x)$ $\forall x, y\,(x + y = y + x)$ $\forall x, y, z\,[(x + y) + z = x + (y + z)]$ $\forall y\,(x + 0 = x \wedge 0 + x = x)$ $\forall x, y, z\,[(x + y) \cdot z = (x \cdot y) + (x \cdot z)]$ $\forall x, y\,[x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)]$	$\mathbb{N}$
Integral domains	$\forall x \exists y\,(x + y = 0)$	$\mathbb{Z}$
Fields	$\forall x\,[x \neq 0 \rightarrow \exists y\,(x \cdot y = 1)]$	$\mathbb{Q}$
Algebraically closed fields	for all $n \geq 1$ $\forall x_0 \ldots x_n \exists y \sum x_i \cdot y^i = 0$	$\mathbb{C}$

Algebraic closure of structures for $\{+, \cdot, 0, 1\}$

Structures	Axioms	Example
Commutative semirings with no zero divisors	$\forall x, y\,(x \cdot y = y \cdot x)$ $\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ $\forall x\,(x \cdot 1 = x \wedge 1 \cdot x = x)$ $\forall x, y\,(x + y = y + x)$ $\forall x, y, z\,[(x + y) + z = x + (y + z)]$ $\forall y\,(x + 0 = x \wedge 0 + x = x)$ $\forall x, y, z\,[(x + y) \cdot z = (x \cdot y) + (x \cdot z)]$ $\forall x, y\,[x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)]$	$\mathbb{N}$
Integral domains	$\forall x \exists y\,(x + y = 0)$	$\mathbb{Z}$
Fields	$\forall x\,[x \neq 0 \rightarrow \exists y\,(x \cdot y = 1)]$	$\mathbb{Q}$
Algebraically closed fields	for all $n \geq 1$ $\forall x_0 \ldots x_n \exists y \sum x_i \cdot y^i = 0$	$\mathbb{C}$

Algebraic closure of structures for $\{+, \cdot, 0, 1\}$

Structures	Axioms	Example
Commutative semirings with no zero divisors	$\forall x, y\,(x \cdot y = y \cdot x)$ $\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ $\forall x\,(x \cdot 1 = x \wedge 1 \cdot x = x)$ $\forall x, y\,(x + y = y + x)$ $\forall x, y, z\,[(x + y) + z = x + (y + z)]$ $\forall y\,(x + 0 = x \wedge 0 + x = x)$ $\forall x, y, z\,[(x + y) \cdot z = (x \cdot y) + (x \cdot z)]$ $\forall x, y\,[x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)]$	$\mathbb{N}$
Integral domains	$\forall x \exists y\,(x + y = 0)$	$\mathbb{Z}$
Fields	$\forall x\,[x \neq 0 \rightarrow \exists y\,(x \cdot y = 1)]$	$\mathbb{Q}$
Algebraically closed fields	for all $n \geq 1$ $\forall x_0 \ldots x_n \exists y \sum x_i \cdot y^i = 0$	$\mathbb{C}$

Algebraic closure of structures for $\{+, \cdot, 0, 1\}$

Structures	Axioms	Example
Commutative semirings with no zero divisors	$\forall x, y\,(x \cdot y = y \cdot x)$ $\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ $\forall x\,(x \cdot 1 = x \wedge 1 \cdot x = x)$ $\forall x, y\,(x + y = y + x)$ $\forall x, y, z\,[(x + y) + z = x + (y + z)]$ $\forall y\,(x + 0 = x \wedge 0 + x = x)$ $\forall x, y, z\,[(x + y) \cdot z = (x \cdot y) + (x \cdot z)]$ $\forall x, y\,[x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)]$	$\mathbb{N}$
Integral domains	$\forall x \exists y\,(x + y = 0)$	$\mathbb{Z}$
Fields	$\forall x\,[x \neq 0 \rightarrow \exists y\,(x \cdot y = 1)]$	$\mathbb{Q}$
Algebraically closed fields	for all $n \geq 1$ $\forall x_0 \ldots x_n \exists y \sum x_i \cdot y^i = 0$	$\mathbb{C}$

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \not\prec_1 \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \prec_1 \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

$$\exists x\,(x^2 - 2 = 0)? \qquad \exists x\,(x^3 + 2x + i = 0)?$$

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \not\prec_1 \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \prec_1 \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

Definition

Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \not\prec_1 \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \prec_1 \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

Definition

Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

- A τ-formula $\phi(x_1, \dots, x_n)$ is quantifier free if it is a boolean combination of atomic formulae.

Existentially closed structures and model companionship

- A τ-formula $\phi(x_1, \ldots, x_n)$ is quantifier free if it is a boolean combination of atomic formulae.

Example

In the vocabulary $\{+, \cdot, 0, 1\}$, the atomic formulae are diophantine equations and the quantifier free formulae with parameters in a ring $\mathcal{M}$ define the constructible sets (in the sense of algebraic geometry) of $\mathcal{M}$:

$$\bigvee_{j=1}^{l} \left[\bigwedge_{i=1}^{k_j} p_{ij}(a_1^{ij}, \ldots, a_{m_{ij}}^{ij}, x_1, \ldots, x_n) = 0 \wedge \bigwedge_{d=1}^{m_j} \neg q_{dj}(b_1^{dj}, \ldots, b_{k_{dj}}^{dj}, x_1, \ldots, x_n) = 0 \right]$$

with each a_k^{ij}, b_k^{dj} elements of $\mathcal{M}$ and
$p_{ij}(y_1, \ldots, y_{m_{ij}}, x_1, \ldots, x_n) = 0$, $q_{dj}(z_1, \ldots, z_{k_{dj}}, x_1, \ldots, x_n) = 0$
diophantine equations (of degree 1 in the y_l, z_h-s).

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1\rangle \sqsubseteq \langle \mathbb{C}, +, \cdot, 0, 1\rangle \sqsubseteq \langle \mathbb{C}[X], +, \cdot, 0, 1\rangle$$

$$\langle \mathbb{Z}, +, \cdot, 0, 1\rangle \not\prec_1 \langle \mathbb{C}, +, \cdot, 0, 1\rangle \prec_1 \langle \mathbb{C}[X], +, \cdot, 0, 1\rangle$$

Definition

Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

- A τ-formula $\phi(x_1, \ldots, x_n)$ is quantifier free if it is a boolean combination of atomic formulae.
- A τ-formula $\psi(x_0, \ldots, x_n)$ is a Σ_1-formula if it is of the form $\exists y_0, \ldots, y_k \phi(y_0, \ldots, y_k, x_0, \ldots, x_n)$ with $\phi(y_0, \ldots, y_k, x_0, \ldots, x_n)$ quantifier free.

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1\rangle \quad \not\prec_1 \quad \langle \mathbb{C}, +, \cdot, 0, 1\rangle \quad \prec_1 \quad \langle \mathbb{C}[X], +, \cdot, 0, 1\rangle$$

Definition

Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

- A τ-formula $\psi(x_0, \dots, x_n)$ is a Σ_1-formula if it is of the form $\exists y_0, \dots, y_k \phi(y_0, \dots, y_k, x_0, \dots, x_n)$ with $\phi(y_0, \dots, y_k, x_0, \dots, x_n)$ quantifier free.

Definition

Given a τ-theory S, a τ-structure $\mathcal{M}$ is S-ec if:

- there is a model of S $\mathcal{N} \sqsupseteq \mathcal{M}$,
- $\mathcal{M} \prec_1 \mathcal{N}$ for any $\mathcal{N} \sqsupseteq \mathcal{M}$ which models S.

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1\rangle \quad \sqsubseteq \quad \langle \mathbb{C}, +, \cdot, 0, 1\rangle \quad \sqsubseteq \quad \langle \mathbb{C}[X], +, \cdot, 0, 1\rangle$$

Definition

Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

Definition

Given a τ-theory S, a τ-structure $\mathcal{M}$ is S-ec if:

- there is a model of S $\mathcal{N} \sqsupseteq \mathcal{M}$,
- $\mathcal{M} \prec_1 \mathcal{N}$ for any $\mathcal{N} \sqsupseteq \mathcal{M}$ which models S.

Example

For S the $\{+, \cdot, 0, 1\}$-theory of integral domains the algebraically closed fields are the S-ec models.

Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1\rangle \quad \not\prec_1 \quad \langle \mathbb{C}, +, \cdot, 0, 1\rangle \quad \prec_1 \quad \langle \mathbb{C}[X], +, \cdot, 0, 1\rangle$$

Definition

Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

Definition

Given a τ-theory S, a τ-structure $\mathcal{M}$ is S-ec if:

- there is a model of S $\mathcal{N} \sqsupseteq \mathcal{M}$,
- $\mathcal{M} \prec_1 \mathcal{N}$ for any $\mathcal{N} \sqsupseteq \mathcal{M}$ which models S.

Example

For S the $\{+, \cdot, 0, 1\}$-theory of integral domains the algebraically closed fields are the S-ec models.

Existentially closed structures and model companionship

Definition

Given a τ-theory S, a τ-structure $\mathcal{M}$ is S-ec if:

- there is a model of S $\mathcal{N} \sqsupseteq \mathcal{M}$,
- $\mathcal{M} \prec_1 \mathcal{N}$ for any $\mathcal{N} \sqsupseteq \mathcal{M}$ which models S.

Definition

Given a τ-theory S, a τ-theory T is the *model companion* of S if TFAE for any τ-structure $\mathcal{M}$:

- $\mathcal{M}$ is a model of T,
- $\mathcal{M}$ is S-ec.

Existentially closed structures and model companionship

Definition

Given a τ-theory S, a τ-structure $\mathcal{M}$ is S-ec if:

- there is a model of S $\mathcal{N} \sqsupseteq \mathcal{M}$,
- $\mathcal{M} \prec_1 \mathcal{N}$ for any $\mathcal{N} \sqsupseteq \mathcal{M}$ which models S.

Definition

Given a τ-theory S, a τ-theory T is the *model companion* of S if TFAE for any τ-structure $\mathcal{M}$:

- $\mathcal{M}$ is a model of T,
- $\mathcal{M}$ is S-ec.

Example

The $\{+, \cdot, 0, 1\}$-theory of integral domains has the $\{+, \cdot, 0, 1\}$-theory of algebraically closed fields as its model companion.

The right vocabulary for a mathematical theory

Every mathematical theory can be axiomatized in first order logic by suitably choosing the vocabulary for its basic concepts.

The right vocabulary for a mathematical theory

Every mathematical theory can be axiomatized in first order logic by suitably choosing the vocabulary for its basic concepts.
Consider Group Theory

The right vocabulary for a mathematical theory

Axioms of groups in $\{\cdot, e\}$

$$\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)],$$
$$\forall y\,(e \cdot y = y \wedge y \cdot e = y),$$
$$\forall x \exists y\,[x \cdot y = e \wedge y \cdot x = e].$$

The right vocabulary for a mathematical theory

Axioms of groups in $\{\cdot, e\}$

$$\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)],$$
$$\forall y\,(e \cdot y = y \wedge y \cdot e = y),$$
$$\forall x \exists y\,[x \cdot y = e \wedge y \cdot x = e].$$

Axioms of groups in $\{R, e\}$ with R a ternary relation symbol

$\forall x, y \exists! z\, R(x, y, z),$

$\forall x, y, z, w, t\,[((R(x, y, w) \wedge R(y, z, t)) \rightarrow \exists u\,(R(x, t, u) \wedge R(w, z, u))],$

$\forall y\,[R(e, y, y) \wedge R(y, e, y)],$

$\forall x \exists y\,[R(x, y, e) \wedge R(y, x, e)].$

The right vocabulary for a mathematical theory

Axioms of groups in $\{\cdot, e\}$

$$\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)],$$
$$\forall y\,(e \cdot y = y \wedge y \cdot e = y),$$
$$\forall x \exists y\,[x \cdot y = e \wedge y \cdot x = e].$$

Axioms of groups in $\{R, e\}$ with R a ternary relation symbol

$\forall x, y \exists! z\, R(x, y, z)$,

$\forall x, y, z, w, t\,[((R(x, y, w) \wedge R(y, z, t)) \rightarrow \exists u\,(R(x, t, u) \wedge R(w, z, u))]$,

$\forall y\,[R(e, y, y) \wedge R(y, e, y)]$,

$\forall x \exists y\,[R(x, y, e) \wedge R(y, x, e)]$.

The two axiomatizions are equivalent in the vocabulary $\{R, \cdot, e\}$, modulo the axiom

$$\forall x, y, z\,(R(x, y, z) \leftrightarrow x \cdot y = z)$$

The right vocabulary for set theory

Standard axiomatization of sets in textbooks is done in vocabulary $\{\in\}$, eventually with extra symbol $\subseteq$.

The right vocabulary for set theory

Standard axiomatization of sets in textbooks is done in vocabulary $\{\in\}$, eventually with extra symbol $\subseteq$.

Formalizing in the $\{\in\}$-vocabulary the notion of ordered pair:

Kuratowski's trick: $\langle y, z \rangle$ is coded in set theory by the set $\{\{y\}, \{y, z\}\}$.

The right vocabulary for set theory

Standard axiomatization of sets in textbooks is done in vocabulary $\{\in\}$, eventually with extra symbol $\subseteq$.

Formalizing in the $\{\in\}$-vocabulary the notion of ordered pair:

Kuratowski's trick: $\langle y, z\rangle$ is coded in set theory by the set $\{\{y\}, \{y, z\}\}$.
In set theory the standard $\in$-formula expressing $x = \langle y, z\rangle$ is

$$\exists t \exists u\, [\forall w\, (w \in x \leftrightarrow w = t \vee w = u) \wedge \forall v\, (v \in t \leftrightarrow v = y) \wedge \forall v\, (v \in u \leftrightarrow v = y \vee v = z)].$$

The right vocabulary for set theory

The vocabulary $\in_{\Delta_0}$ for set theory

- constants for $\emptyset, \mathbb{N}$,
- relation symbols R_ϕ for any lightface Δ_0-property $\phi(x_1, \ldots, x_n)$,
- function symbols for a finite list of basic set theoretic constructors.

The right vocabulary for set theory

Lightface Δ_0-properties

- $\{R \in V : R \text{ is an } n\text{-ary relation}\}$,
- $\{f \in V : f \text{ is a function}\}$,
- $\left\{\langle a, b\rangle \in V^2 : a \subseteq b\right\}$,
- ...
- $\{\langle a_1, \ldots, a_n\rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$ for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set.

The right vocabulary for set theory

Lightface Δ_0-properties

- $\{R \in V : R$ is an n-ary relation$\}$,
- $\{f \in V : f$ is a function$\}$,
- $\left\{\langle a, b\rangle \in V^2 : a \subseteq b\right\}$,
- ...
- $\{\langle a_1, \ldots, a_n\rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$ for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set (e.g. $y \subseteq z \equiv \forall x\,(x \in y \rightarrow x \in z) \equiv \forall x \in y\,(x \in z)$).

The *lightface Δ_0-properties* are those described in the last item above and include all those listed in some of the above items.

The right vocabulary for set theory

Lightface Δ_0-properties

- $\{R \in V : R$ is an n-ary relation$\}$,
- $\{f \in V : f$ is a function$\}$,
- $\left\{\langle a, b\rangle \in V^2 : a \subseteq b\right\}$,
- ...
- $\{\langle a_1, \ldots, a_n\rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$ for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set.

Complicated set theoretic relations

- $\left\{\langle X, Y\rangle \in V^2 : |X| = |Y|\right\}$,
- $\left\{\langle X, Y\rangle \in V^2 : X = \mathcal{P}(Y)\right\}$,
- ...
- any relation which is not a Δ_1-property

The right vocabulary for set theory

Complicated set theoretic relations

- $\{\langle X, Y\rangle \in V^2 : |X| = |Y|\}$,
- $\{\langle X, Y\rangle \in V^2 : X = \mathcal{P}(Y)\}$,
- $\ldots$
- any relation which is not a Δ_1-property $(\Delta_0 \subseteq \Delta_1)$.

The right vocabulary for set theory

Basic set theoretic operations

- $\pi^n_j : \langle a_1, \ldots, a_n \rangle \mapsto a_j$,
- $\langle X, Y \rangle \mapsto X \times Y$,
- $\langle X, Y \rangle \mapsto \{X, Y\}$,
- $\ldots$
- Any provably total function whose graph is a lightface Δ_0-property.

The right vocabulary for set theory

The vocabulary $\in_{\Delta_0}$ for set theory

- constants for $\emptyset, \mathbb{N}$,
- relation symbols R_ϕ for any lightface Δ_0-property $\phi(x_1, \ldots, x_n)$,
- function symbols for a finite list of basic set theoretic constructors.

Basic set theoretic operations

- $\pi^n_j : \langle a_1, \ldots, a_n \rangle \mapsto a_j$,
- $\langle X, Y \rangle \mapsto X \times Y$,
- $\langle X, Y \rangle \mapsto \{X, Y\}$,
- $\ldots$
- Any provably total function whose graph is a lightface Δ_0-property.

The right vocabulary for set theory

The vocabulary $\in_{\Delta_0}$ for set theory

- constants for $\emptyset, \mathbb{N}$,
- relation symbols R_ϕ for any lightface Δ_0-property $\phi(x_1, \ldots, x_n)$,
- function symbols for a finite list of basic set theoretic constructors.

Lightface Δ_0-properties

$$\{\langle a_1, \ldots, a_n \rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$$

for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set.

Basic set theoretic operations

Any total function whose graph is a lightface Δ_0-property.

Section 6

Formalization of set theory

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\mathrm{Set}, V\}$

Notational convention: smallcase variables indicate sets, uppercase variables indicate classes.

Universal axioms

- **Extensionality:** $\forall X, Y\,[(X \subseteq Y \wedge Y \subseteq X) \leftrightarrow X = Y]$.
- **Comprehension:** $\forall X\,(\mathrm{Set}(X) \leftrightarrow X \in V) \wedge \forall X\,(X \subseteq V)$.
- **Foundation:**

$$\forall F\,[(F \text{ is a function } \wedge \mathrm{dom}(F) = \mathbb{N}) \rightarrow \exists n \in \mathbb{N}\, F(n+1) \notin F(n)].$$

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Existence Axioms:

- **Emptyset:** $(\forall x\, x \notin \emptyset) \wedge (\emptyset \in V)$,
- **Infinity:**
  $\text{Set}(\mathbb{N}) \wedge \forall x\, [x \in \mathbb{N} \leftrightarrow (x \text{ is a finite Von Neumann ordinal})]$.

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Basic construction principles:

- **Union and Pair:** $\forall X, Y, w\,[w \in X \cup Y \leftrightarrow (w \in X \vee w \in Y)], \ldots$
- **Separation:** $\forall P, x\,[(x \in V) \rightarrow (P \cap x) \in V]$.

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\mathrm{Set}, V\}$

Strong construction principles:

- **Comprehension (b):** For every $\in_{\Delta_0}$-formula $\psi(\vec{x}, \vec{Y})$

$$\forall \vec{Y} \exists Z \forall x \, [x \in Z \leftrightarrow (x \in V \wedge \exists x_0, \ldots, x_n (x = \langle x_0, \ldots, x_n \rangle \wedge \psi(x_0, \ldots, x_n, \vec{Y})))].$$

- **Replacement:**

$$\forall F, x \, [(F \text{ is a function } \wedge (x \in V) \wedge (x \subseteq \mathrm{dom}(F))) \rightarrow (F[x] \in V)].$$

- **Powerset:**

$$\forall x \, [(x \in V) \rightarrow [\forall z \, (z \in \mathcal{P}(X) \leftrightarrow z \subseteq x) \wedge \mathcal{P}(x) \in V]].$$

- **Choice:**

$$\forall F[\; F \text{ is a function} \wedge \forall x \, (x \in \mathrm{dom}(F) \rightarrow F(x) \neq \emptyset) \rightarrow \exists G \, (G \text{ is a function} \wedge \mathrm{dom}(G) = \mathrm{dom}(F) \wedge \forall x \, (x \in \mathrm{dom}(G) \rightarrow G(x) \in F(x)) \;].$$

Section 7

Algebraic maximality for set theory

The H_κs

A finite set may not be simple, for example to understand the singleton $\{\mathbb{R}\}$ we need to know $\mathbb{R}$.

The H_κs

A finite set may not be simple, for example to understand the singleton $\{\mathbb{R}\}$ we need to know $\mathbb{R}$.

Definition

A set X is *hereditarily finite* if it is finite and all its elements are finite, and all the elements of its elements are finite,. . .

The H_κs

A finite set may not be simple, for example to understand the singleton $\{\mathbb{R}\}$ we need to know $\mathbb{R}$.

Definition

A set X is *hereditarily finite* if it is finite and all its elements are finite, and all the elements of its elements are finite, . . . ,
i.e. if letting

- $\bigcup^0 X = X$,
- $\bigcup^{n+1} X = \bigcup(\bigcup^n X)$,
- $\mathrm{trcl}(X) = \bigcup_{n\in\mathbb{N}}(\bigcup^n X)$,

$\mathrm{trcl}(X)$ is finite.

The H_κs

Definition

A set X is *hereditarily finite* if it is finite and all its elements are finite, and all the elements of its elements are finite,. . . ,
i.e. if letting

- $\bigcup^0 X = X$,
- $\bigcup^{n+1} X = \bigcup(\bigcup^n X)$,
- $\mathrm{trcl}(X) = \bigcup_{n \in \mathbb{N}}(\bigcup^n X)$,

$\mathrm{trcl}(X)$ is finite.

Example

- $\{\mathbb{R}\}$ is not hereditarily finite;
- each $n \in \mathbb{N}$ is hereditarily finite (recall that $n = \{0, \dots, n-1\}$ for all $n \in \mathbb{N}$);

The H_κs

Definition

A set X is *hereditarily finite* if $\mathtt{trcl}(X)$ is finite.

$H_{\aleph_0}$ is the set of all hereditarily finite sets.

The H_κs

Definition

A set X is *hereditarily finite* if $\mathtt{trcl}(X)$ is finite.

$H_{\aleph_0}$ is the set of all hereditarily finite sets.

Definition

A set X is *hereditarily countable* if $\mathtt{trcl}(X)$ is countable.

$H_{\aleph_0^+} = H_{\aleph_1}$ is the set of all hereditarily countable sets.

The H_κs

Definition

A set X is *hereditarily finite* if $\mathtt{trcl}(X)$ is finite.
$H_{\aleph_0}$ is the set of all hereditarily finite sets.

Definition

A set X is *hereditarily countable* if $\mathtt{trcl}(X)$ is countable.
$H_{\aleph_0^+} = H_{\aleph_1}$ is the set of all hereditarily countable sets.

Remark

- $\{\mathbb{R}\}$ *is not hereditarily countable;*
- *Any subset of* $\mathbb{N}$ *is hereditarily countable;*
- $\mathbb{Q}$ *and* $\mathbb{Z}$ *as defined in any textbook are hereditarily countable;*
- $\mathbb{R}$ *and* $\mathcal{P}(\mathbb{N})$ *are subsets of* $H_{\aleph_1}$ *(but not elements!);*
- $\mathcal{P}(\mathbb{N})$ *is definable by the atomic* $\in_{\Delta_0}$*-formula* $(x \subseteq \mathbb{N})$ *in the structure* $\langle H_{\aleph_1}, \in_{\Delta_0}\rangle$*;*
- *similarly for* $\mathbb{R}$ *or for any Polish space.*

The H_κs

Definition

A set X is *hereditarily finite* if $\mathtt{trcl}(X)$ is finite.
$H_{\aleph_0}$ is the set of all hereditarily finite sets.

Definition

A set X is *hereditarily countable* if $\mathtt{trcl}(X)$ is countable.
$H_{\aleph_0^+} = H_{\aleph_1}$ is the set of all hereditarily countable sets.

Definition

Given a cardinal κ, a set X is *hereditarily of size at most* κ if $\mathtt{trcl}(X)$ has size at most κ;
H_{κ^+} is the set of all sets which are hereditarily of size at most κ.

The H_κs

Definition

A set X is *hereditarily finite* if $\mathtt{trcl}(X)$ is finite.
$H_{\aleph_0}$ is the set of all hereditarily finite sets.

Definition

A set X is *hereditarily countable* if $\mathtt{trcl}(X)$ is countable.
$H_{\aleph_0^+} = H_{\aleph_1}$ is the set of all hereditarily countable sets.

Definition

$H_{\aleph_1^+} = H_{\aleph_2}$ is the set of all sets which are hereditarily of size at most $\aleph_1$.

The H_κs

Definition

A set X is *hereditarily countable* if $\mathtt{trcl}(X)$ is countable.
$H_{\aleph_0^+} = H_{\aleph_1}$ is the set of all hereditarily countable sets.

Definition

$H_{\aleph_1^+} = H_{\aleph_2}$ is the set of all sets which are hereditarily of size at most $\aleph_1$.

Remark

- $\mathcal{P}(\aleph_1)$ *is definable by the atomic* $\in_{\Delta_0}$*-formula* $(x \subseteq \aleph_1)$ *in parameter* $\aleph_1$ *(the first uncountable ordinal) in the structure* $\langle H_{\aleph_2}, \in_{\Delta_0}\rangle$*,*
- NS*, the non-stationary ideal on* $\aleph_1$*, is* Σ_1*-definable in parameter* $\aleph_1$ *in the same structure.*

The H_κs

Definition

A set X is *hereditarily countable* if $\mathrm{trcl}(X)$ is countable.
$H_{\aleph_0^+} = H_{\aleph_1}$ is the set of all hereditarily countable sets.

Definition

Given a cardinal κ, a set X is *hereditarily of size at most* κ if $\mathrm{trcl}(X)$ has size at most κ;
H_{κ^+} is the set of all sets which are hereditarily of size at most κ.

Definition

$H_{\aleph_1^+} = H_{\aleph_2}$ is the set of all sets which are hereditarily of size at most $\aleph_1$.

$$H_{\aleph_0} \subseteq H_{\aleph_1} \subseteq H_{\aleph_2} \subseteq \cdots \subseteq H_{\kappa^+} \subseteq \ldots$$

$$V = \bigcup \{H_{\kappa^+} : \kappa \text{ an infinite cardinal}\}$$

Existentially closed structures for set theory

Theorem (Levy)

Let κ be an infinite cardinal.
Then

$$\langle H_{\kappa^+}, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle \prec_1 \langle V, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

Theorem (Levy)

Let κ be an infinite cardinal.
Then

$$\langle H_{\kappa^+}, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle \prec_1 \langle V, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle.$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

Theorem (Levy)

Let κ be an infinite cardinal.
Then

$$\langle H_{\kappa^+}, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle \prec_1 \langle V, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle.$$

Theorem (Shoenfield, 1961)

Let $V[G]$ be a forcing extension of V. Then

$$\langle H_{\aleph_1}, \in_{\Delta_0}\rangle \prec_1 \langle V[G], \in_{\Delta_0}\rangle.$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- (modulo a Borel isomorphism) $\mathbb{R} \approx \mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$ and UB is a family of subsets of $\mathcal{P}(\mathbb{N})$.
- Every univ. Baire set A of V can be naturally lifted to a univ. Baire set $A^{V[G]}$ of $V[G]$ for any forcing extension $V[G]$ of V.

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- (modulo a Borel isomorphism) $\mathbb{R} \approx \mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$ and UB is a family of subsets of $\mathcal{P}(\mathbb{N})$.
- Every univ. Baire set A of V can be naturally lifted to a univ. Baire set $A^{V[G]}$ of $V[G]$ for any forcing extension $V[G]$ of V.

Theorem (Feng-Magidor-Woodin, 1992)

Let $V[G]$ be a forcing extension of V. Then

$$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle \prec_1 \langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle.$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

Theorem (Levy)

Let κ be an infinite cardinal.
Then

$$\langle H_{\kappa^+}, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle \prec_1 \langle V, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle.$$

Theorem (Shoenfield, 1961)

Let $V[G]$ be a forcing extension of V. Then

$$\langle H_{\aleph_1}, \in_{\Delta_0}\rangle \prec_1 \langle V[G], \in_{\Delta_0}\rangle.$$

Theorem (Feng-Magidor-Woodin, 1992)

Let $V[G]$ be a forcing extension of V. Then

$$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V\rangle \prec_1 \langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V\rangle.$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- (modulo a Borel isomorphism) $\mathbb{R} \approx \mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$ and UB is a family of subsets of $\mathcal{P}(\mathbb{N})$.
- Every univ. Baire set A of V can be naturally lifted to a univ. Baire set $A^{V[G]}$ of $V[G]$ for any forcing extension $V[G]$ of V.

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- (modulo a Borel isomorphism) $\mathbb{R} \approx \mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$ and UB is a family of subsets of $\mathcal{P}(\mathbb{N})$.
- Every univ. Baire set A of V can be naturally lifted to a univ. Baire set $A^{V[G]}$ of $V[G]$ for any forcing extension $V[G]$ of V.

Theorem (Woodin, 1985+Martin-Steel, 1989+ V.-Venturi, 2020)

Assume there is a proper class of Woodin's cardinals. Then the theory of

$$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathsf{UB}^V \rangle$$

is the ***model companion*** *of the theory of*

$$\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathsf{UB}^V \rangle$$

for any forcing extension $V[G]$ of V.

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is Σ_1-elementary in $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V
MK+ large cardinal axioms	The theory of $\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is the **model companion** of the theory of $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is Σ_1-elementary in $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V
MK+ large cardinal axioms	The theory of $\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is the **model companion** of the theory of $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V

Stationary sets and the non-stationary ideal on $\aleph_1$

Definition

- C is a club subset of $\aleph_1$ if $\sup(C) = \aleph_1$ and for all $\beta \notin C$ there is $\alpha < \beta$ such that $[\alpha, \beta] \cap C$ is empty.
- $S \subseteq \aleph_1$ is stationary if for all C club subset of $\aleph_1$ $S \cap C$ is non-empty.
- $\mathsf{NS} \subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$ (i.e. subsets disjoint from some club).
- NS is saturated if the boolean algebra $\mathcal{P}(\aleph_1)/_{\mathsf{NS}}$ has only partitions of size at most $\aleph_1$.

Theorem

- *Assume* NS *is saturated. Then it is precipitous.*
- *Assume Martin's Maximum* MM. *Then* NS *is saturated.*
- NS *is precipitous is consistent with* CH.

Stationary sets and the non-stationary ideal on $\aleph_1$

Definition

- C is a club subset of $\aleph_1$ if $\sup(C) = \aleph_1$ and for all $\beta \notin C$ there is $\alpha < \beta$ such that $[\alpha, \beta] \cap C$ is empty.
- $S \subseteq \aleph_1$ is stationary if for all C club subset of $\aleph_1$ $S \cap C$ is non-empty.
- $\mathrm{NS} \subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$ (i.e. subsets disjoint from some club).
- NS is saturated if the boolean algebra $\mathcal{P}(\aleph_1)/_{\mathrm{NS}}$ has only partitions of size at most $\aleph_1$.

Theorem

- *Assume* NS *is saturated. Then it is precipitous.*
- *Assume Martin's Maximum* MM. *Then* NS *is saturated.*
- NS *is precipitous is consistent with* CH.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *I*

- NS $\subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part I

- NS $\subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Definition

Let B be a cba. B is SSP if whenever $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V \rangle \sqsubseteq \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]} \rangle.$$

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part I

- NS $\subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Definition

Let B be a cba. B is SSP if whenever $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V\rangle \sqsubseteq \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}\rangle.$$

Definition

Strong Bounded Martin's maximum BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}\rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}\rangle.$$

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *I*

- NS $\subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Definition

Let B be a cba. B is SSP if whenever $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V\rangle \sqsubseteq \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}\rangle.$$

Definition

Strong Bounded Martin's maximum BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}\rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}\rangle.$$

Theorem (Bagaria, Woodin)

MM^{++} *implies* BMM^{++}.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *I*

Definition

Let B be a cba. B is SSP if whenever $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \text{NS}^V \rangle \sqsubseteq \langle V[G], \in_{\Delta_0}, \text{NS}^{V[G]} \rangle.$$

Definition

Strong Bounded Martin's maximum BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \text{NS} \rangle \prec_1 \langle V[G], \in_{\Delta_0}, \text{NS}^{V[G]} \rangle.$$

Theorem (Bagaria, Woodin)

MM^{++} *implies* BMM^{++}.

MM^{++} is consistent with the existence of any axiom of large cardinals.

Applications of BMM^{++}

Assume BMM^{++}. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^+$.
  Todorčević, Mathematical Research Letters, 9(2), 2006.
- Whitehead's conjecture on free groups is false,
  (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  All automorphisms of the Calkin algebra are inner.

Applications of BMM^{++}

Assume BMM^{++}. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^+$.
  Todorčević, Mathematical Research Letters, 9(2), 2006.
- Whitehead's conjecture on free groups is false,
  (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  All automorphisms of the Calkin algebra are inner.

Applications of BMM^{++}

Assume BMM^{++}. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^+$.
  Todorčević, Mathematical Research Letters, 9(2), 2006.
- Whitehead's conjecture on free groups is false,
  (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  All automorphisms of the Calkin algebra are inner.

Applications of BMM^{++}

Assume BMM^{++}. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^+$.
  Todorčević, Mathematical Research Letters, 9(2), 2006.
- Whitehead's conjecture on free groups is false,
  (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  All automorphisms of the Calkin algebra are inner.

Applications of BMM^{++}

Assume BMM^{++}. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^+$.
  Todorčević, Mathematical Research Letters, 9(2), 2006.
- Whitehead's conjecture on free groups is false,
  (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  All automorphisms of the Calkin algebra are inner.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- $\mathrm{NS} \subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- $\mathsf{NS} \subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- $\mathrm{NS} \subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Definition (Woodin-Schindler?)

UB-BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$$

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- $\mathrm{NS} \subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

Definition (Woodin-Schindler?)

$\mathrm{UB}\text{-}\mathrm{BMM}^{++}$ holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$$

Theorem (Woodin)

MM^{++} *implies* $\mathrm{UB}\text{-}\mathrm{BMM}^{++}$.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

Definition (Woodin-Schindler?)

UB-BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$$

Theorem (Woodin)

MM^{++} *implies* UB-BMM^{++}.

$(*)_{\mathrm{UB}}$ is a natural strengthening of Woodin's axiom $(*)$.

Theorem (Asperó-Schindler)

Assume there is a proper class of Woodin cardinals. Then $(*)_{\mathrm{UB}}$ *if and only if* UB-BMM^{++}.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

Definition (Woodin-Schindler?)

UB-BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$$

Theorem (Woodin)

MM^{++} *implies* UB-BMM^{++}.

$(*)_{\mathrm{UB}}$ is a natural strengthening of Woodin's axiom $(*)$.

Theorem (Asperó-Schindler)

Assume there is a proper class of Woodin cardinals. Then $(*)_{\mathrm{UB}}$ *if and only if* UB-BMM^{++}.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

Definition (Woodin-Schindler?)

UB-BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$$

Theorem (Asperó-Schindler)

Assume there is a proper class of Woodin cardinals. Then Woodin's axiom $(*)$ *holds if and only if whenever* B *is an SSP cba and* $V[G]$ *is a forcing extension of* V *by* B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \text{ is in } \mathcal{P}(\mathbb{R})^{L(\mathbb{R})^V} \rangle$$

is Σ_1*-elementary in*

$$\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \text{ is in } \mathcal{P}(\mathbb{R})^{L(\mathbb{R})^V} \rangle.$$

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

In signature $\in_{\Delta_0}$ $\neg$CH can be formalized by the Π_2-sentence in parameter $\aleph_1$ (the first uncountable ordinal/cardinal):

$$\forall f\,[\underbrace{f \text{ is a function}}_{\Delta_0(f)} \wedge \underbrace{\mathrm{dom}(f)=\aleph_1}_{\Delta_0(f,\aleph_1)}) \to \exists r\,(\underbrace{r\subseteq\mathbb{N}}_{\Delta_0(r,\mathbb{N})} \wedge \underbrace{r\notin\mathrm{ran}(f)}_{\Delta_0(r,f)})]$$

Note that $\aleph_1 \in H_{\aleph_2}$.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

Theorem (Woodin)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *For any* Π_2*-sentences* ψ *for* $\in_{\Delta_0} \cup \{\aleph_1, \mathrm{NS}\} \cup \left\{A : A \in \mathrm{UB}^V\right\}$

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle \models \psi$$

if and only if

ψ is true in $H_{\aleph_2}^{V[G]}$ for some forcing extension $V[G]$ of V.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (Woodin)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* $\mathrm{UB\text{-}BMM}^{++}$*).*
- *For any* Π_2*-sentences* ψ *for* $\in_{\Delta_0} \cup \{\aleph_1, \mathrm{NS}\} \cup \left\{A : A \in \mathrm{UB}^V\right\}$ *(among which* $\neg\mathrm{CH}$ *and a strong form of* $2^{\aleph_0} = \aleph_2$*)*

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \models \psi$$

if and only if

ψ is true in $H_{\aleph_2}^{V[G]}$ for some forcing extension $V[G]$ of V.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (Woodin)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *For any* Π_2*-sentences* ψ *for* $\in_{\Delta_0} \cup \{\aleph_1, \mathrm{NS}\} \cup \left\{A : A \in \mathrm{UB}^V\right\}$

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \models \psi$$

if and only if

ψ is true in $H_{\aleph_2}^{V[G]}$ for some forcing extension $V[G]$ of V.

Sealing can be removed by replacing UB with $\mathcal{P}(\mathbb{R})^N$ for some nice inner model N of determinacy in the formulation of BMM^{*++} and in the relevant spots.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *The theory T of the structure*

$$\mathcal{M} = \langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$$

*is the **model companion** of the theory S of the structure*

$$\langle V, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle.$$

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *The theory T of the structure*

$$\mathcal{M} = \langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$$

*is the **model companion** of the theory S of the structure*

$$\langle V, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle.$$

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *The theory T of the structure*

$$\mathcal{M} = \langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$$

*is the **model companion** of the theory S of the structure*

$$\langle V, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle.$$

- *Letting $S_{\forall\vee\exists}$ be the boolean combination of existential sentences which are in S, and ψ be a Π_2-sentence,*
  $\mathcal{M}$ ***models*** ψ ***if and only*** $\psi + S_{\forall\vee\exists}$ ***is consistent***.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *For any* Π_2*-sentences* ψ

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \models \psi$$

if and only if

ψ *is true in* $H_{\aleph_2}^{V[G]}$ *for some forcing extension* $V[G]$ *of* V*.*

if and only if

$\psi + S_{\forall\vee\exists}$ *is consistent*

where S *is the theory of the structure*

$$\langle V, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle.$$

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathsf{UB}}$ *(or* UB-BMM^{++}*).*
- *The theory T of the structure*

$$\mathcal{M} = \langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathsf{NS}, A : A \in \mathsf{UB}^V \rangle$$

  *is the **model companion** of the theory S of the structure*

$$\langle V, \in_{\Delta_0}, \aleph_1, \mathsf{NS}, A : A \in \mathsf{UB}^V \rangle.$$

- *Letting $S_{\forall\vee\exists}$ be the boolean combination of existential sentences which are in S, and ψ be a Π_2-sentence,*
  $\mathcal{M}$ models ψ if and only $\psi + S_{\forall\vee\exists}$ is consistent.

Sealing can be removed if one replaces UB^V with $\mathcal{P}(R)^{L(\mathrm{Ord}^{\mathbb{N}})}$ in the formulation of BMM^{*++} and in the relevant spots.

Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle$ is a *substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ forcing axioms	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle$ is a Σ_1-*substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ large cardinal axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ have the same **model companion** theory
MK+ large cardinals + forcing axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ have as **model companion** the theory of $\langle H^V_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V, A^V : A \in \mathrm{UB}^V\rangle$

Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle$ is a *substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ forcing axioms	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle$ is a Σ_1-*substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ large cardinal axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ have the same **model companion** theory
MK+ large cardinals + forcing axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ have as **model companion** the theory of $\langle H^V_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V, A^V : A \in \mathrm{UB}^V\rangle$

Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle$ is a *substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ forcing axioms	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle$ is a Σ_1-*substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ large cardinal axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ have the same **model companion** theory
MK+ large cardinals + forcing axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V\rangle$ have as **model companion** the theory of $\langle H^V_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V, A^V : A \in \mathrm{UB}^V\rangle$

Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$ is a *substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ forcing axioms	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$ is a Σ_1-*substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ large cardinal axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ have the same **model companion** theory
MK+ large cardinals + forcing axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ have as **model companion** the theory of $\langle H^V_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V, A^V : A \in \mathrm{UB}^V \rangle$

Section 8

Appendixes

Appendix 1: Sealing

Definition (Woodin)

Given $(\mathcal{D}, W, \in_{\Delta_0})$ transitive model of MK, let N^W be the set $\mathcal{P}(H_{\aleph_1})^{L(\mathrm{UB})^W}$, where $L(\mathrm{UB})^W$ is the smallest transitive model of ZF containing UB^W.

(A weak form of) Sealing holds in a model $(\mathcal{C}, V, \in_{\Delta_0})$ of MK+*enough large cardinals* if whenever $V[G]$ is a forcing extension of V and $V[H]$ a forcing extension of $V[G]$ we have that

$$(N^{V[G]}, H_{\aleph_1}^{V[G]}, \in_{\Delta_0}) \prec (N^{V[H]}, H_{\aleph_1}^{V[H]}, \in_{\Delta_0}).$$

Theorem (Woodin)

Assume V models κ is supercompact and there are class many Woodin cardinals. Let $V[H]$ be a generic extension of V where κ is countable. Then sealing holds in $V[H]$.

Appendix 1: Sealing

Definition (Woodin)

Given $(\mathcal{D}, W, \in_{\Delta_0})$ transitive model of MK, let N^W be the set $\mathcal{P}(H_{\aleph_1})^{L(\mathrm{UB})^W}$, where $L(\mathrm{UB})^W$ is the smallest transitive model of ZF containing UB^W.

(A weak form of) Sealing holds in a model $(\mathcal{C}, V, \in_{\Delta_0})$ of MK+*enough large cardinals* if whenever $V[G]$ is a forcing extension of V and $V[H]$ a forcing extension of $V[G]$ we have that

$$(N^{V[G]}, H_{\aleph_1}^{V[G]}, \in_{\Delta_0}) \prec (N^{V[H]}, H_{\aleph_1}^{V[H]}, \in_{\Delta_0}).$$

Theorem (Woodin)

Assume V models κ is supercompact and there are class many Woodin cardinals. Let $V[H]$ be a generic extension of V where κ is countable. Then sealing holds in $V[H]$.

Appendix 1: Sealing

Definition (Woodin)

Given $(\mathcal{D}, W, \in_{\Delta_0})$ transitive model of MK, let N^W be the set $\mathcal{P}(H_{\aleph_1})^{L(\mathrm{UB})^W}$, where $L(\mathrm{UB})^W$ is the smallest transitive model of ZF containing UB^W.
(A weak form of) Sealing holds in a model $(\mathcal{C}, V, \in_{\Delta_0})$ of MK+*enough large cardinals* if whenever $V[G]$ is a forcing extension of V and $V[H]$ a forcing extension of $V[G]$ we have that

$$(N^{V[G]}, H_{\aleph_1}^{V[G]}, \in_{\Delta_0}) \prec (N^{V[H]}, H_{\aleph_1}^{V[H]}, \in_{\Delta_0}).$$

Theorem (Woodin)

Assume V models κ is supercompact and there are class many Woodin cardinals. Let $V[H]$ be a generic extension of V where κ is countable. Then sealing holds in $V[H]$.

Appendix 2: Some references

A few surveys on Gödel's program and the Continuum problem:

- J. Bagaria, *Natural axioms on set theory and the continuum problem*, CRM Preprint, 591, 2004.
- P. Koellner, *On the question of absolute undecidability*, in *Kurt Gödel: essays for his centennial*, Lect. Notes Log. 33, 2010.
- G. Venturi and M. Viale, *What model companionship can say about the Continuum problem*, arXiv:2204.13756, 2022.
- M. Viale, *Strong forcing axioms and the continuum problem*, in *Séminaire Bourbaki. Volume 2022/2023. Exposés 1197–1211*, 2023, (SMF).
- W. H. Woodin, *The Continuum hypothesis Part I*, Notices of AMS, 48(6), 2001.
- W. H. Woodin, *The Continuum hypothesis Part II*, Notices of AMS, 48(7), 2001.