# Lecture 1: The classics

## Model theory of fields
- algebraic questions
- model-theoretic methods
- often: algebraic answers.

## §1. The power of compactness

for an analytic fct $f: \mathbb{C}^n \to \mathbb{C}^n$, this follows from Picard's Thm.

### §1.1: Ax's Theorem (aka Ax-Grothendieck)

Every injective polynomial map $f: \mathbb{C}^n \to \mathbb{C}^n$ is surjective

i.e. $f = (g_1(X_1, \dots, X_n), \dots, g_n(X_1, \dots, X_n))$
$\in K[X_1, \dots, X_n]$

Why? Consider $\mathbb{C}$ in $\mathcal{L}_{ring} = \{0, 1, +, \cdot\}$

Then $\mathbb{C} \models ACF = \{\text{field axioms}\}$
$\cup \{\forall y_0, \dots, y_{n-1} \exists x \; x^n + \sum y_i x^i = 0 : n > 1\}$

"Fundamental Theorem of Algebra"

Tarski: The completions of ACF are given by

$ACF_p = ACF \cup \{\underbrace{1 + \dots + 1}_{p\text{-times}} = 0\}$ for p prime

and $ACF_0 = ACF \cup \{\neg \underbrace{1 + \dots + 1}_{p\text{-times}} = 0 : p \text{ prime}\}$

Lefschetz principle: An $\mathcal{L}_{ring}$-sentence $\varphi$ holds in almost all $\mathbb{F}_p^{alg}$ iff it holds in $\mathbb{C}$.

Pf: $\mathbb{C} \models \varphi \iff ACF_0 \models \varphi \iff ACF_0 \vdash \varphi$
$\iff ACF_p \vdash \varphi$ for almost all p
$\iff \mathbb{F}_p^{alg} \models \varphi$ -"-

For fixed $n, m$ let $\varphi_{n,m}$ be the $\mathcal{L}_{ring}$-sentence "every inj. polynomial map $f: K^n \to K^n$ of degree $\leq m$ is surjective"

WTS: $\mathbb{C} \models \varphi_{n,m}$. Show instead: $\mathbb{F}_p^{alg} \models \varphi_{n,m}$ $\forall p$.

Lemma 1: $F$ finite field. If $f: F^n \to F^n$ is inj., then $f$ is surjective [Pf: obvious]

Lemma 2: If $f: (\mathbb{F}_p^{alg})^n \to (\mathbb{F}_p^{alg})^n$ is an injective polynomial map, then $f$ is surjective.

Pf: Take $m_0 \in \mathbb{N}$ s.th. all coeff. of $f$ are in $\mathbb{F}_{p^{m_0}}$

If $m \geq m_0$: $f \restriction_{\mathbb{F}_{p^m}}: (\mathbb{F}_{p^m})^n \to (\mathbb{F}_{p^m})^n$ is inj., hence surjective

As $\mathbb{F}_p^{alg} = \bigcup_{m \geq m_0} \mathbb{F}_{p^m}$, $f$ is surjective. $\boxtimes$

## §1.2: A theorem by Ax-Kochen and Ershov

Consider $\mathbb{Q}_p = \{ \sum_{i=m}^{\infty} a_i p^i : m \in \mathbb{Z}, a_i \in \{0, \dots, p-1\} \}$

and $\mathbb{F}_p((t)) = \{ \sum_{i=m}^{\infty} a_i t^i : m \in \mathbb{Z}, a_i \in \{0, \dots, p-1\} \}$, in $\mathcal{L}_{ring}$.

Ax-Kochen/Ershov Theorem (Version 1) For any $\mathcal{L}_{ring}$-sentence $\varphi$ ex. $N(\varphi) \in \mathbb{N}$ s.th. $\forall p > N(\varphi)$:

$\mathbb{Q}_p \models \varphi \iff \mathbb{F}_p((t)) \models \varphi$.

(more details soon)

Artin's Conjecture: $\mathbb{Q}_p$ is $C_2$, i.e. every homogeneous polynomial of degree $d$ in $d^2$-many variables has a non-trivial root.

Lang: $\mathbb{F}_p((t))$ is $C_2$ for all $p$.

AK/E: for a fixed degree $d$, $\mathbb{Q}_p$ is $C_2(d)$ for $p \gg 0$.

Terjanian: No $\mathbb{Q}_p$ is $C_2$.

$\hookrightarrow$ Artin's Conjecture holds asymptotically, but fails nonetheless.

AIM OF TALK 1: Explain the AK/E Theorem.
AIM OF TALK 2: Remove "asymptotically".

## §2: Valuations & Infinitesimals

Def: A valuation on a field $K$ is a map $v: K \twoheadrightarrow \Gamma \cup \{\infty\}$ where $\Gamma$ is an ordered abelian group, s.th. $\forall x, y \in K$:

- $v(x) = \infty \iff x = 0$
- $v(x \cdot y) = v(x) + v(y)$
- $v(x+y) \geq \min(v(x), v(y))$

$\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$ valuation ring "bounded elements"

$\mathfrak{m}_v = \{x \in K : v(x) > 0\}$ maximal ideal "infinitesimals"

$Kv := \mathcal{O}_v / \mathfrak{m}_v$ residue field, $vK := \Gamma$ value group.

Examples: $p$ prime

$x \in \mathbb{Q}^\times$, $x = p^d a/b$ with $a,b \in \mathbb{Z}$, $p \nmid a,b$ $\Rightarrow v_p(x) := d$

$x \in \mathbb{F}_p(t)^\times$, $x = t^d a/b$ with $a,b \in \mathbb{F}_p[t]$, $t \nmid a,b$ $\Rightarrow v_t(x) := d$.

$\rightsquigarrow$ similar for $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

e.g. $v_p\left(\sum_{i=m}^{\infty} a_i p^i\right) = \min\{i : a_i \neq 0\}$

Here: $vK \cong \mathbb{Z}$, $Kv = \mathbb{F}_p$

$v_p(p) = 1$ $\qquad$ $v_t(t) = 1$.

What's the difference?

$\mathbb{F}_p((t))$, $\mathbb{Q}_p$ are complete, $\mathbb{F}_p(t)$ and $\mathbb{Q}$ are not.

Def: $(K,v)$ is henselian if for all $f \in \mathcal{O}_v[X]$ and $b \in \mathcal{O}_v$ with $f(b) \in \mathfrak{m}_v$ infinitesimal and $f'(b) \notin \mathfrak{m}_v$ there is a $\beta \in \mathcal{O}_v$ with $f(\beta) = 0$ and $\beta - b \in \mathfrak{m}_v$.

infinitesimally close

Thm (Hensel's Lemma) If $(K,v)$ with $vK \leq \mathbb{R}$ is complete, then $v$ is henselian (in particular: $(\mathbb{Q}_p, v_p)$, $(\mathbb{F}_p((t)), v_t)$)

Proof: via Newton approximation.

Language for valued fields (version 1)

$\mathcal{L}_{val} = \mathcal{L}_{ring} \cup \{\mathcal{O}\}$

$\rightsquigarrow$ $\mathfrak{m}_v = \{x \in \mathcal{O}_v : x^{-1} \notin \mathcal{O}_v\}$, $Kv = \mathcal{O}_v/\mathfrak{m}_v$

$vK = K^\times/\mathcal{O}_v^\times$ $\qquad$ all interpretable

**Thm (AK/E, version 2, in $\mathcal{L}_{val}$)** Let $(K,v)$, $(L,w)$ be henselian valued fields with $\mathrm{char}(Kv) = \mathrm{char}(Lw) = 0$. Then

$$(K,v) \equiv_{\mathcal{L}_{val}} (L,w) \iff Kv \equiv_{\mathcal{L}_{ring}} Lw \text{ and } vK \equiv_{\underset{\{0,+,<\}}{\mathcal{L}_{oag}}} wL$$

**Corollary**: Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{P}$. Then

$$\prod_{\mathcal{U}} (\mathbb{Q}_p, v_p) \equiv_{\mathcal{L}_{val}} \prod_{\mathcal{U}} (\mathbb{F}_p((t)), v_t)$$

In particular, AK/E version 1 holds.

**Pf**: Let $(K,v)$ denote either $\prod_{\mathcal{U}} (\mathbb{Q}_p, v_p)$ or $\prod_{\mathcal{U}} (\mathbb{F}_p((t)), v_t)$. Then

$$Kv \cong \prod_{\mathcal{U}} \mathbb{F}_p \quad \text{and} \quad vK \cong \prod_{\mathcal{U}} \mathbb{Z}$$

Note that $(K,v)$ is henselian and $\mathrm{char}(Kv) = 0$. it's $\mathcal{L}_{val}$-elem. !

Assuming (CH), $(K,v)$ is $\aleph_1$-sat. of size $\aleph_1$, so we even get $\cong$ (via back-and-forth) $\boxtimes$

**Corollary**: $(K,v)$ henselian valued field, $\mathrm{char}(Kv) = 0$. Then $(K,v) \equiv (k((\Gamma)), v_\Gamma)$ for $k = Kv$ and $\Gamma = vK$.

"up to elementary equivalence, every hens. eq.char 0 field is a power series field"

## §3: How about a proof? or at least a sketch...

### Language for valued fields (version 2)

Consider a three-sorted language: sorts $\mathbb{K}, \Gamma, \mathbb{k}$

$\mathcal{L}_{Pas} = \mathcal{L}_{ring}$ on sort $\mathbb{K}$, $\mathcal{L}_{oag,\infty} = \{+,-,<,0,\infty\}$
on sort $\Gamma$, $\mathcal{L}_{ring}$ on sort $\mathbb{k}$
maps: $v: \mathbb{K} \to \Gamma$, $ac: \mathbb{K} \to \mathbb{k}$

**Def:** An ac-map is a map $ac: K \to Kv$ s.th.

- $ac(0) = 0$
- $ac \restriction_{K^\times} : K^\times \to Kv^\times$ mult. grp. homom
- $ac(x) = res(x)$ for all $x \in \mathcal{O}^\times$
  "$x + \mathfrak{m}_v \in Kv$"

In $\mathbb{Q}_p$: $ac\left(\sum_{i=m}^{\infty} a_i p^i\right) := a_{\min\{i : a_i \neq 0\}}$
(same in $\mathbb{F}_p((t))$)

**Note:** Not every valued field admits an ac-map
(but suff. sat. ones do)
If $s: \Gamma \to K$ section of $v$, set $ac(x) := res\left(\frac{x}{s(x)}\right)$

Given an o.a.g. $\Gamma$ and a field $k$ with $char(k) = 0$,
consider the $\mathcal{L}_{Pas}$-theory
$T_{k,\Gamma}$ = " any model $(K,v)$ is henselian with
$Kv \equiv_{\mathcal{L}_{ring}} k$, $vK \equiv_{\mathcal{L}_{oag}} \Gamma$ and

$$\pi(x) := \begin{cases} ac(x) & \text{if } v(x) = 0 \\ 0 & \text{otherwise} \end{cases}$$

is a surj. ring homom $\mathcal{O}_v \to \mathbb{k}$, $\ker(\pi) = \mathfrak{m}_v$."

**Pas' Theorem**: $T_{k,\Gamma}$ eliminates $\mathbb{K}$-quantifiers in $\mathcal{L}_{Pas}$.

i.e. for $\Sigma = \{ \mathcal{L}_{Pas}$ - fmlae with no quantifiers over $\mathbb{K} \}$,
any $\mathcal{L}_{Pas}$ - fmla is modulo $T_{k,\Gamma}$ equiv. to a formula in $\Sigma$.

Use (a variant of) Shoenfield's criterion
$T$ $\mathcal{L}$-theory, $\Sigma$ set of $\mathcal{L}$-fmlae closed under B.C.
Sps for some $\kappa > |T|$, for any $M, N \models T$ $\kappa$-sat,
for any $f : A \xrightarrow{\sim} B$ partial isom preserving $\Sigma$,
$|A| < \kappa$ $\quad M^{\subseteq} \quad {}^{\subseteq} N$
and any $a \in M \setminus A$, $f$ extends to
$f' : A' \xrightarrow{\sim} B'$ with $a \in A'$, $|A'| < \kappa$
preserving $\Sigma$. Then any $\mathcal{L}$-fmla is equivalent, mod $T$, to one in $\Sigma$.

**More details?**

See Simone Ramello's excellent walkthrough here:

https://www.uni-muenster.de/IVV5WS/WebHop/user/sramello/ake-slides.pdf