Introduction to continuum theory and projective Fraïssé theory

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Projective Fraïssé theory – setup

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2. A topological L-structure is a compact zero-dimensional second-countable space A equipped with closed relations $R_i^A, i \in I$ and continuous functions $f_j^A, j \in J$.
3. Epimorphisms are continuous surjections preserving the structure.

Projective Fraïssé class – definition

A family $\mathcal{F}$ of finite topological L-structure is a projective Fraïssé class if:

1. (F1) (joint projection property: JPP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and epimorphisms from C onto A and from C onto B;
2. (F2) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any epimorphisms $\phi_1 \colon B_1 \to A$ and $\phi_2 \colon B_2 \to A$, there exist C, $\phi_3 \colon C \to B_1$ and $\phi_4 \colon C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

Projective Fraïssé limit – definition

A topological L-structure $\mathbb{L}$ is a projective Fraïssé limit of $\mathcal{F}$ if the following three conditions hold:

1. (L1) (projective universality) for any $A \in \mathcal{F}$ there is an epimorphism from $\mathbb{L}$ onto A;
2. (L2) (projective ultrahomogeneity) for any $A \in \mathcal{F}$ and any epimorphisms $\phi_1 \colon \mathbb{L} \to A$ and $\phi_2 \colon \mathbb{L} \to A$ there exists an isomorphism $h \colon \mathbb{L} \to \mathbb{L}$ such that $\phi_2 = \phi_1 \circ h$;
3. (L3) for any finite discrete topological space X and any continuous function $f \colon \mathbb{L} \to X$ there is an $A \in \mathcal{F}$, an epimorphism $\phi \colon \mathbb{L} \to A$, and a function $f_0 \colon A \to X$ such that $f = f_0 \circ \phi$.

Projective Fraïssé limit – existence and uniqueness

Theorem (Irwin-Solecki)

Let $\mathcal{F}$ be a countable projective Fraïssé class of finite structures. Then:

1. *there exists a projective Fraïssé limit of $\mathcal{F}$;*
2. *any two projective Fraïssé limits are isomorphic.*

A simple example of a projective Fraïssé class

Let $\mathcal{F}$ be the family of all finite sets.

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Then the projective Fraïssé limit is the Cantor set.

One more simple example

Let $\mathcal{F}$ be the family of all finite sets $A = \{a_1, \ldots, a_n\}$, some n, with the binary relation $\leq^A$, where for each i, $a_i \leq^A a_i$ and $a_i \leq^A a_{i+1}$.

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Then the projective Fraïssé limit is $(\mathbb{C}, \leq^{\mathbb{C}})$, where $\mathbb{C}$ is the Cantor set. For $a \neq b \in \mathbb{C}$, we have $a \leq^{\mathbb{C}} b$ or $b \leq^{\mathbb{C}} a$ iff a and b are endpoints of an interval removed at some stage of the construction of $\mathbb{C}$, viewed as the middle-third Cantor set.

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Identify $\leq^{\mathbb{C}}$-related points. This is the *topological realization* of $(\mathbb{C}, \leq^{\mathbb{C}})$. It is homeomorphic to [0,1].

The pseudo-arc

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A continuum is indecomposable if it is not the union of two proper subcontinua.

Definition

It is hereditarily indecomposable if its every subcontinuum is indecomposable.

Definition

A continuum is chainable if any open cover can be refined by an open cover $U_1, \ldots, U_n$ such that for all $i, j \leq n$, we have $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.

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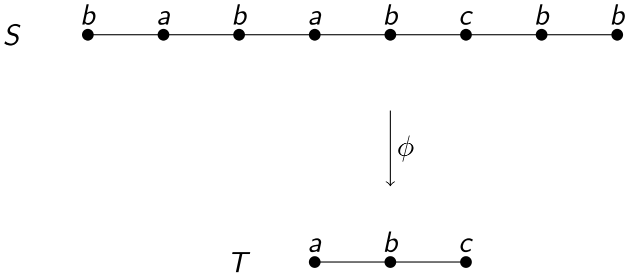
A continuous surjection $\phi\colon S \to T$ is an epimorphism iff

$$r^T(a, b) \iff \exists c, d \in S \left(\phi(c) = a, \phi(d) = b, \text{ and } r^S(c, d) \right).$$

An example of an epimorphism



Construction of the pseudo-arc, part 2

Theorem (Irwin-Solecki)

1. *The family $\mathcal{G}$ has the amalgamation property.*
2. *There is a unique $\mathbb{P} = (\mathbb{P}, r^{\mathbb{P}})$, where $\mathbb{P}$ is compact, separable, totally disconnected, $r^{\mathbb{P}}$ is closed, which is projectively universal, projectively ultrahomogeneous, and continuous maps onto finite sets factor through epimorphisms onto finite structures.*
3. *The relation $r^{\mathbb{P}}$ is an equivalence relation such that each equivalence class has at most two elements.*

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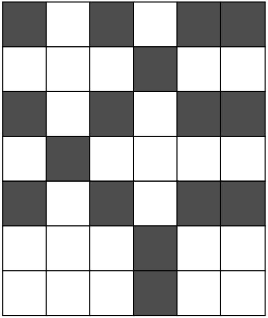
Theorem (Irwin-Solecki)

$\mathbb{P}/r^{\mathbb{P}}$ is the pseudo-arc.

Amalgamation property

Theorem (Steinhaus chessboard theorem)

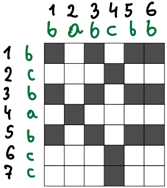
Consider a chessboard $m \times n$ with some squares black and some white. Assume that the king cannot go across the chessboard from the left edge to the right moving exclusively on black squares. Then the rook can go across the chessboard from upper edge to the lower one moving exclusively on white squares.

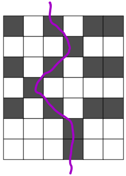


Amalgamation property 2

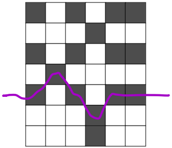
$f: \underbrace{babcbb}_{B} \to \underbrace{abc}_{A}$

$g: \underbrace{bcbabcc}_{C} \to abc$





$\{(x,y): f(x)=g(y)\}$ = black



π_i - projections

$$\begin{tikzcd} & D \arrow[dl, dashed, "\pi_1"'] \arrow[dr, dashed, "\pi_2"] & \\ B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

D - combine both purple paths into one

$D=\{(1,3),(2,4),(3,3),(4,2),(5,3),(6,4),(7,4),(6,4),(5,5),(5,6),(5,5),(6,4),(5,3),(4,2),(5,1)\}$

The projective universality and homogeneity of $\mathbb{P}$ yield the following theorem.

Theorem

(i) *(Mioduszewski) Each chainable continuum is a continuous image of the pseudo-arc.*

(ii) *(Irwin-Solecki) Let X be a chainable continuum with a metric d on it. If f_1, f_2 are continuous surjections from the pseudo-arc onto X, then for any $\epsilon > 0$ there exists a homeomorphism h of the pseudo-arc such that $d(f_1(x), f_2 \circ h(x)) < \epsilon$ for all x.*

Knaster continua

Definition

A Knaster continuum is a continuum homeomorphic to the inverse limit $\varprojlim(I_n, f_n)$ of a sequence of unit intervals $I_n = [0, 1]$ with continuous, open, non-homeomorphic surjections f_n that map 0 to 0.

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- Universal Knaster continuum K is a Knaster continuum which continuously and openly surjects onto all Knaster continua.

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- Universal Knaster continuum K is a Knaster continuum which continuously and openly surjects onto all Knaster continua.
- S. Iyer (2022) constructed the universal Knaster continuum as the topological realization of a projective Fraïssé limit.
- She represented $Homeo(K)$ as the semidirect product of an extremely amenable Polish group and the free abelian group on countably many generators, and concluded that the universal minimal flow of $Homeo(K)$ is non-metrizable.
- Another construction of the universal Knaster continuum in the projective Fraïssé theoretic framework was presented by L. Wickman (2022).

More examples

Example

1. (Bartošová-Kwiatkowska '15) Lelek fan
   $\mathcal{F} = \{\text{rooted trees, all epimorphisms}\}$
2. (Panagiotopoulos-Solecki '22) Menger universal curve
   $\mathcal{F} = \{\text{finite connected graphs, monotone epimorphisms}\}$
3. (Charatonik-Roe '22+) Ważewski dendrite W_3
   $\mathcal{F} = \{\text{finite trees, monotone epimorphisms}\}$
4. (Codenotti-Kwiatkowska '22+) all generalized Ważewski dendrites W_P, $P \subseteq \{3, 4, \ldots, \omega\}$
   $\mathcal{F}_P = \{\text{finite trees, weakly coherent monotone epimorphisms}\}$
5. (Bartoš-Kubiś '22+) P-adic pseudo-solenoids for any set of primes P

Topological graphs

Definition

A topological graph K is a graph $(V(K), E(K))$, whose domain $V(K)$ is a 0-dimensional, compact, second-countable (thus has a metric) space and $E(K)$ is a closed, reflexive and symmetric subset of $V(K)^2$.

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Definition

1. A continuous function $f : L \to K$ is a homomorphism if $\langle a, b\rangle \in E(L)$ implies $\langle f(a), f(b)\rangle \in E(K)$.
2. A homomorphism f is an epimorphism if it is moreover surjective on both vertices and edges.

Definition

A subset S of a topological graph G is disconnected if there are two nonempty closed subsets P and Q of S such that $P \cup Q = S$ and if $a \in P$ and $b \in Q$, then $\langle a, b \rangle \notin E(G)$. A subset S of G is connected if it is not disconnected.

Monotone maps

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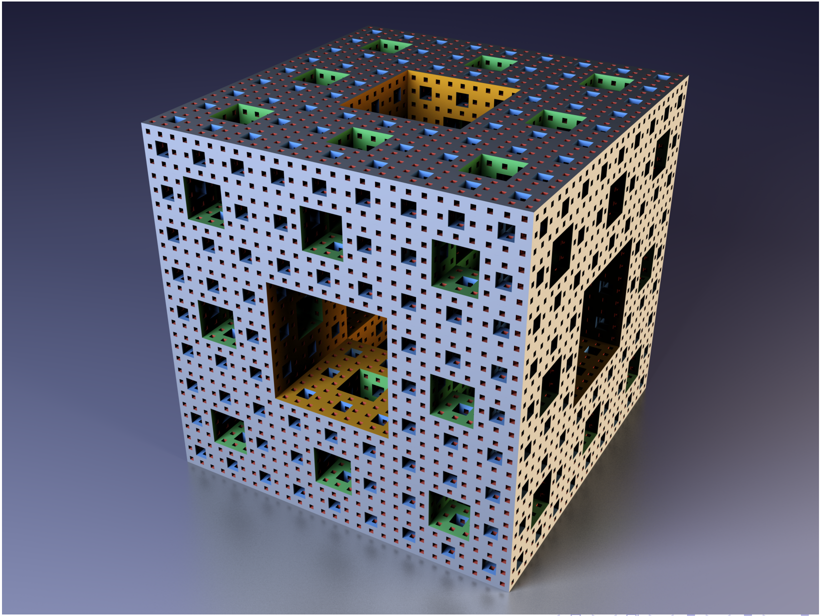
Definition

- (continua) Let K, L be continua. A continuous map $f\colon L \to K$ is called monotone if for every subcontinuum M of K, $f^{-1}(M)$ is connected.
- (graphs) Let G, H be topological graphs. An epimorphism $f\colon G \to H$ is called monotone if for every closed connected subset Q of H, $f^{-1}(Q)$ is connected.

Universal Menger curve - construction



Universal Menger curve - Fraïssé construction

Theorem (Panagiotopoulos-Solecki)

The class $\mathcal{F}$ of all finite connected graphs with monotone epimorphisms is a Fraïssé class. The topological realization of the projective Fraïssé limit of $\mathcal{F}$ is the universal Menger curve.

Let $\mathbb{M}$ denote the projective Fraïssé limit of $\mathcal{F}$.

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Theorem (Panagiotopoulos-Solecki '22)

1. *Each Peano continuum is a continuous monotone image of the universal Menger curve.*
2. *Let X be a Peano continuum. Let d be a metric on X. If f_1 and f_2 are continuous monotone surjections from the universal Menger curve onto X, then for any $\epsilon > 0$ there exists a homeomorphism h of the universal Menger curve such that for all x, $d(f_1(x), f_2 \circ h(x)) < \epsilon$.*

Homogeneity of the universal Menger curve

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A topological subgraph K of $\mathbb{M}$ is locally non-separating if for each clopen connected W, the set $W \setminus K$ is connected.

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If K and L are saturated and locally non-separating subgraphs of $\mathbb{M}$, then each isomorphism from K to L extends to an automorphism of $\mathbb{M}$.

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Corollary (Anderson '58)

Any bijection between finite subsets of the universal Menger curve extends to a homeomorphism.

Confluent maps

Definition

- (continua) Let K, L be continua. A continuous map $f\colon L \to K$ is called confluent if for every subcontinuum M of K and every component C of $f^{-1}(M)$ we have $f(C) = M$.
- (graphs) Let G, H be topological graphs. An epimorphism $f\colon G \to H$ is called confluent if for every closed connected subset Q of H and every component C of $f^{-1}(Q)$ we have $f(C) = Q$.

More on confluent maps

Proposition (Charatonik-Roe '22+)

Given two finite graphs G and H, the following conditions are equivalent for an epimorphism $f: G \to H$:

1. *f is confluent;*
2. *for every edge $P \in E(H)$ and every component C of $f^{-1}(P)$ there is an edge E in C such that $f(E) = P$.*

Fraïssé classes of cycles

Definition

For $A \in \mathcal{G}$ we will say that $C \subseteq A$ is a cycle in A if $|V(C)| > 2$ and we can enumerate the vertices of C as $(c_0, c_1, \ldots, c_n = c_0)$ in a way that $c_i \neq c_j$ whenever $0 \leq i < j < n$ and $\langle c_i, c_j \rangle \in E(A)$ if and only if $|j - i| \leq 1$.

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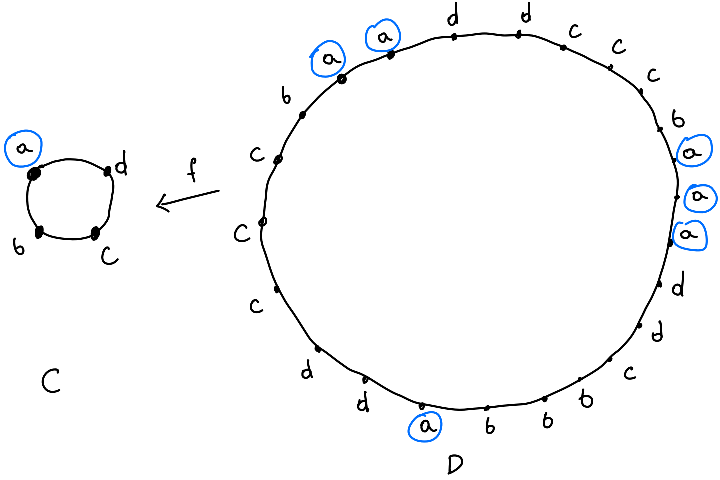
Definition

The winding number of a wrapping map f is n if for every (equivalently: some) $c \in C$, $f^{-1}(c)$ has exactly n components.

Wrapping maps



Constructing solenoids - warm up

Theorem (Charatonik-K-Roe '22+)

Let P be a set of prime numbers and let $\mathcal{D}_P$ be the class of cycles with confluent epimorphisms whose winding numbers are of the form $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where $p_i \in P$ and $n_i \in \mathbb{N}$. Then $\mathcal{D}_P$ is a projective Fraïssé class.
The projective Fraïssé limit is the solenod, which is surjectively universal in the class of solenoids constructed using winding numbers from P.

The projective Fraïssé class $\mathcal{G}$ - main example

Proposition (Charatonik-Roe '22+)

The class $\mathcal{G}$ of finite connected graphs with confluent epimorphisms is a projective Fraïssé class.

- Let $\mathbb{G}$ denote the projective Fraïssé limit. Then $E(\mathbb{G})$ is an equivalence relation with only single and double equivalence classes.
- Let $|\mathbb{G}|$ denote the topological realization. This is a one-dimensional continuum.

Main Theorem - part 1

Theorem (Charatonik-K-Roe '22+)

$|\mathbb{G}|$ *has the following properties:*

1. *it is not homogeneous;*
2. *it is pointwise self-homeomorphic;*
3. *it is an indecomposable continuum;*
4. *all arc components are dense;*
5. *each point is the top of the Cantor fan;*
6. *the pseudo-arc, the universal pseudo-solenoid, and the universal solenoid, embed in it;*
7. *it is hereditarily unicoherent, in particular, the circle S^1 does not embed in it.*

Main theorem - part 2: Embedding solenoids and non-homogeneity

Theorem (Charatonik-K-Roe '22+)

There is a dense set of points in $|\mathbb{G}|$ that belong to a solenoid. Moreover, the only solenoid that embeds into $|\mathbb{G}|$ is the universal solenoid.

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Corollary (Charatonik-K-Roe '22+)

The continuum $|\mathbb{G}|$ is not homogeneous.

Thank you!