Introduction to continuum theory and projective Fraïssé theory

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Chainable continua 1

Definition

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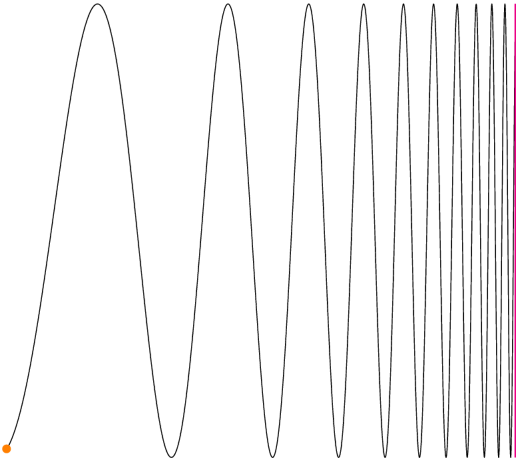
An open cover $\mathcal{U}$ *refines* an open cover $\mathcal{V}$ if every set from $\mathcal{U}$ is contained in some set in $\mathcal{V}$.

Definition

A continuum is chainable if any open cover can be refined by an open cover $U_1, \ldots, U_n$ such that for all $i, j \leq n$, we have $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.

$sin(\frac{1}{x})$-continuum is chainable



Chainable continua 2

Definition

A continuum X is arc-like if for every ϵ, there is a continuous and surjective $f\colon X \to [0,1]$ such that $f^{-1}(t)$ has diameter $< \epsilon$ for every $t \in [0,1]$.

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Theorem

A continuum is chainable iff it is arc-like iff it is an inverse limit of arcs [0,1] with continuous surjective bonding maps.

Indecomposable continua

Definition

A continuum is indecomposable if it is not the union of two proper subcontinua.

Definition

It is hereditarily indecomposable if its every subcontinuum is indecomposable.

Examples of indecomposable continua

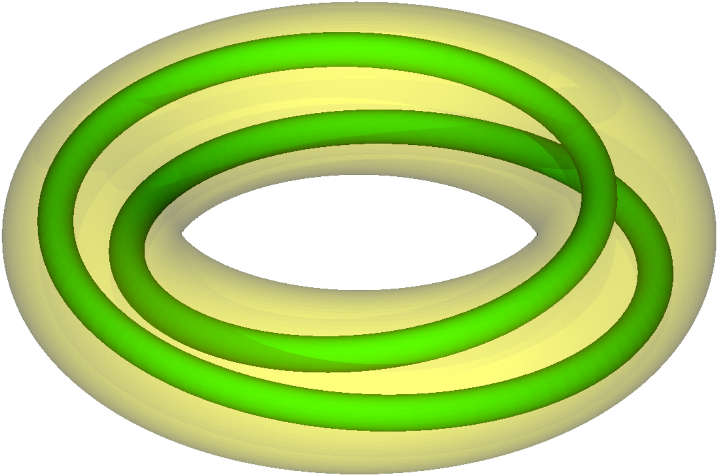
Proposition

Let $X = \varprojlim\{X_i, f_i\}$ be the inverse limit of continua. Suppose that for each i, whenever A_{i+1} and B_{i+1} are subcontinua of X_{i+1} such that $X_{i+1} = A_{i+1} \cup B_{i+1}$, then $f_i(A_{i+1}) = X_i$ or $f_i(B_{i+1}) = X_i$. Then X is indecomposable.

Example (p-adic solenoids)

Let p be a prime number. Let $X_i = \mathbb{S}^1$ be a circle and let $f_i(z) = z^p$. Then the obtained inverse limit is the p-adic solenoid and it is indecomposable.

solenoid, $p = 2$



Solenoids-characterizations

Theorem

Let X be a continuum not homeomorphic to a circle. The following are equivalent:

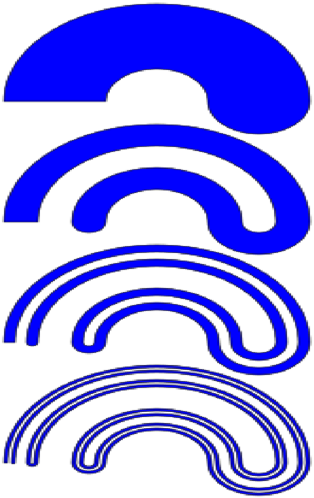
1. *X is a solenoid,*
2. *(Hewitt '63) X is homeomorhic to a one-dimensional topological group,*
3. *(Hagopian '77) X is homogeneous and every proper subcontinuum is an arc.*

A topological space X is homogeneous if for any $a, b \in X$ there is a homeomorphism h of X such that $h(a) = b$.

Knaster continuum is indecomposable

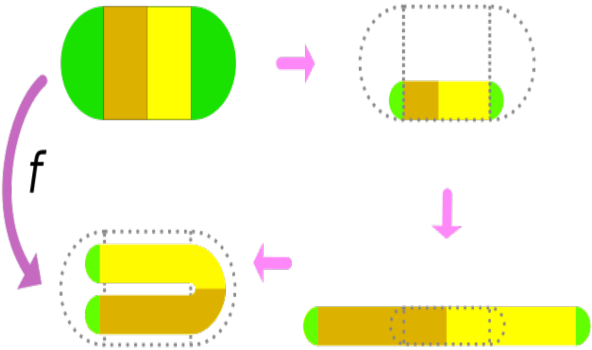
Let $X_i = [0,1]$
and take $X = \varprojlim\{X_i, f_i = f\}$,
where

$$f(t) = \begin{cases} 2t & \textit{for } t \in [0, \frac{1}{2}] \\ -2t + 2 & \textit{for } t \in [\frac{1}{2}, 1] \end{cases}$$



Smale's horseshoe map

- The attractor of this map is the Knaster continuun from above.



The pseudo-arc

Definition

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There is a unique pseudo-arc up to homeomorphism.

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Theorem (Bing '48)

The pseudo-arc P is homogeneous, that is, for any $a, b \in P$ there is a homeomorphism h of P such that $h(a) = b$.

The pseudo-arc 2

Classification of topologically homogeneous plane continua:

Theorem (Hoehn-Oversteegen, 2016)

Up to homeomorphism, the only nondegenerate homogeneous planar continua are

(a) *the circle,*

(b) *the pseudo-arc, and*

(c) *the circle of pseudo-arcs.*

Peano continua

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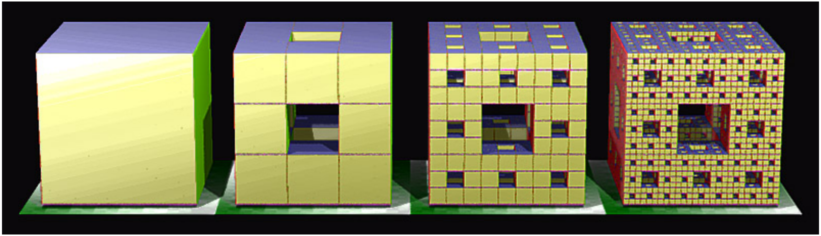
- (Hahn-Mazurkiewicz) Every Peano continuum is a continuous image of the interval [0,1]
- Every Peano continuum is locally arcwise connected.

Universal Menger curve - construction



Menger sponge = universal Menger curve

Theorem (Menger '26)

The universal Menger curve is universal in the class of all metric separable spaces of dimension ≤ 1.

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Theorem (Anderson '58)

The following are equivalent for a continuum X.

1. *X is homeomorphic to the universal Menger curve,*
2. *X is a locally connected curve with no local cut points and no planar open nonempty subsets,*
3. *X is a homogeneous locally connected curve, which is not homeomorphic to a circle.*

Dendrites

Definition

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- Therefore any dendrite is a Peano continuum.
- For any $P \subseteq \{3, 4, 5, \ldots, \omega\}$ there is a unique Ważewski dendrite, that is, a dendrite which has ramification points of orders exactly in P and any arc contains a point of each order from P.

The universal Ważewski dendrite

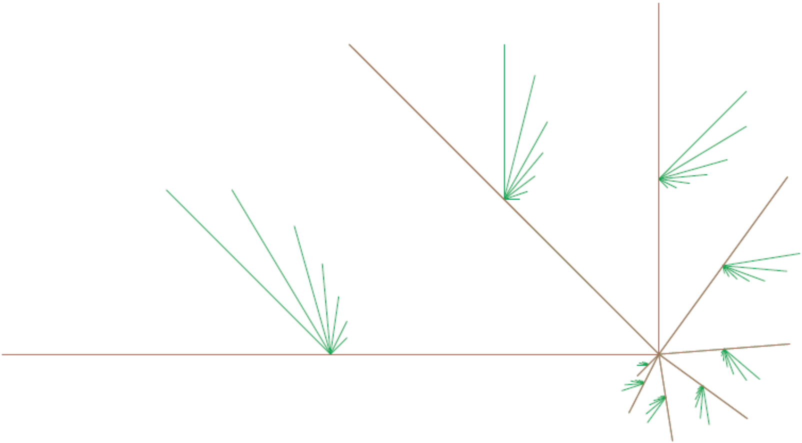

Figure: The universal Ważewski dendrite W_ω